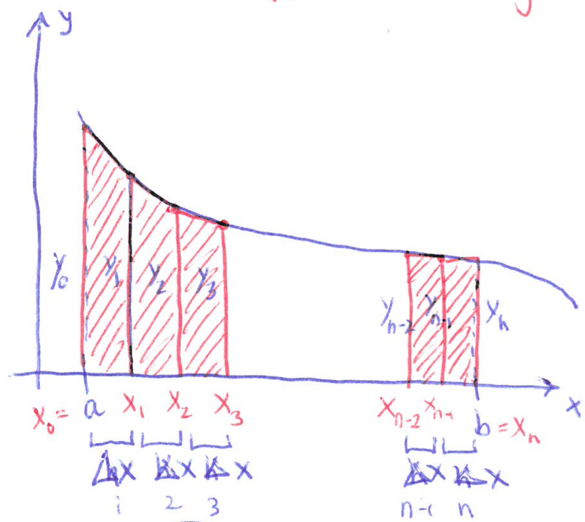
Calc AB

Lesson 95

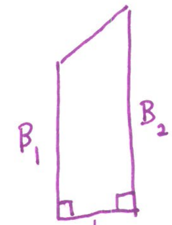
Agenda:
- Lesson 95
  - Trapezoidal Rule
  - Error bound for Trapezoid Rule

Back to Approximating Area under Curves, Now with Trapezoids.



Recall: Area of a Trapezoid

$$\boxed{A = \frac{1}{2}(B_1 + B_2)\cdot h}$$



$$\text{Area} \approx \frac{1}{2}(y_0+y_1)\Delta x + \frac{1}{2}(y_1+y_2)\Delta x + \frac{1}{2}(y_2+y_3)\Delta x + \cdots + \frac{1}{2}(y_{n-1}+y_n)\cdot\Delta x$$

$$= \frac{\Delta x}{2}\left[y_0 + 2y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-1} + y_n\right]$$

Trapezoid Rule

$$T = \boxed{\frac{b-a}{2n}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)}$$

Ex 95.2 Approximate $\int_{3\pi/2}^{2\pi} \frac{\cos x}{x}\,dx$ using the trapezoid rule with $n=4$.

x	$\frac{3\pi}{2}$	$\frac{13\pi}{8}$	$\frac{7\pi}{4}$	$\frac{15\pi}{8}$	2π
y	0	0.075	0.1286	0.1568	0.1592

$$\Delta x = \frac{2\pi - \frac{3\pi}{2}}{4} = \frac{\pi}{8}$$

$$\int_{3\pi/2}^{2\pi} \frac{\cos x}{x}\,dx \approx \frac{1}{2}\left(\frac{\pi}{8}\right)\left(0 + 2(0.075) + 2(0.1286) + 2(0.1568) + 0.1592\right)$$

$$\approx \boxed{0.1728}$$

Error Bound for Trapezoid Rule:

Derivation - Need Integration by Parts

$$\boxed{|E_T| = \left|\int_a^b f(x)\,dx - T\right| \le \frac{(b-a)^3}{12n^2}\cdot \max|f''(x)|}$$

Absolute value of error

Approximation w/ Trapezoid Rule

95.4 Determine the number of subintervals needed to guarantee a trapezoidal rule approx. of $\int_1^3 (x^3 - x)\,dx$ with an error less than 10^{-3}.

Find n with $|E| < 10^{-3}$ since $|E| \le \frac{(3-1)^3}{12n^2}\cdot \max|f''(x)| < 10^{-3}$ So $\boxed{n=110}$

$f''(x) = 6x$ $|f''(x)| \le 18$ on $[1,3]$ so $\frac{\cancel{8}^{2}}{\cancel{12}n^2}\cdot \cancel{18}^{6} < \frac{1}{10^3} \Rightarrow n^2 > 12{,}000$

$n > \sqrt{12000} \approx 109.5$

Estimate $\int_a^b f(x)dx$ using Trapezoid Rule, Prove Error $|E| \le \frac{(b-a)^3}{12n^2} \max|f''(x)|$.

$h = \frac{b-a}{h}$ and $x_i = a + ih$

$$\int_{x_i}^{x_{i+1}} f(x)dx = \int_0^h f(t+x_i)\,dt \qquad \begin{bmatrix} u = f(t+x_i) & v = (t+A) \\ du = f'(t+x_i)\cdot dt & dv = 1dt \end{bmatrix}$$

$$= (t+A)f(t+x_i)\Big|_0^h - \int_0^h (t+A) f'(t+x_i)\,dt$$

$$= (t+A)f(t+x_i)\Big|_0^h - \left[\left(f'(t+x_i)\cdot\left(\frac{t^2}{2}+At+B\right)\right)\Big|_0^h - \int_0^h \left(\frac{t^2}{2}+At+B\right) f''(t+x_i)\,dt\right]$$

A, B are constants - choose so error bound is small

$$= \Big((h+A)f(h+x_i) - Af(x_i)\Big) - \left(f'(h+x_i)\left(\frac{h^2}{2}+Ah+B\right) - f'(x_i)B\right) + \int_0^h \left(\frac{t^2}{2}+At+B\right) f''(t+x_i)\,dt$$

Pick $A = -\frac{h}{2}$ then:

$$= h\left(\frac{f(h+x_i) - f(x_i)}{2}\right) - \Big(f'(h+x_i)B - f'(x_i)B\Big) + \int_0^h \left(\frac{t^2}{2} - \frac{ht}{2} + B\right) f''(t+x_i)\,dt$$

To make the error between $\int_{x_i}^{x_{i+1}} f(x)dx$ and $\frac{h}{2}(f(h+x_i) - f(x_i))$ small, take $B = 0$:

$$= \frac{h}{2}\Big(f(h+x_i) - f(x_i)\Big) + \int_0^h \left(\frac{t^2-ht}{2}\right) f''(t+x_i)\,dt$$

Thus the error

$$|E_{T_i}| = \left|\int_{x_i}^{x_{i+1}} f(x)dx - \frac{h}{2}(f(h+x_i) - f(x_i))\right| = \left|\int_0^h \left(\frac{t^2-ht}{2}\right) f''(t+x_i)\,dt\right|$$

$$\le \int_0^h \left|\left(\frac{t^2-ht}{2}\right)\right| \max|f''(x)|\,dt = \max|f''(x)| \cdot \int_0^h \left|\left(\frac{t^2}{2} - \frac{ht}{2}\right)\right| dt$$

$$= \max|f''(x)| \left(\frac{3ht^2}{12} - \frac{2t^3}{12}\right)\Big|_0^h = \frac{h^3}{12} \max|f''(x)|$$

$$= \frac{(b-a)^3}{12n^3} \max|f''(x)|$$

Therefore $|E_T| \le n \cdot |E_{T_i}| = \frac{(b-a)^3}{12n^2} \max|f''(x)|$ ∎