Review Practice: Chapter 14

Formulas:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$L(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$\mathbf{D_u} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

Second Derivative Test: $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$

Lagrange Multipliers: $\nabla f(x, y, z) = \lambda \nabla g(x_0, y_0, z_0), \quad g(x, y, z) = k$

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y}, \qquad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

1. True or False: If $(2, 1)$ is a critical point of f and $f_{xx}(2, 1) f_{yy}(2, 1) < f_{xy}(2, 1)^2$ then f has a saddle point at $(2, 1)$.

$\boxed{\text{True}}$ $D(2,1) = f_{xx}(2,1) f_{yy}(2,1) - f_{xy}(2,1)^2$

< 0

Thus by Second Derivative Test $(2,1)$ is a saddle point.

2. Find the partial derivatives of $G(x, y, z) = e^{xy} \sin(y/z)$.

$$G_x = \frac{\partial}{\partial x}\left(e^{xy} \sin(y/z)\right) = \boxed{e^{xy}(y)\sin(y/z)}$$

$$G_y = \frac{\partial}{\partial y}(e^{xy}) \sin(y/z) + e^{xy} \frac{\partial}{\partial y}(\sin(y/z))$$

$$= \boxed{e^{xy}(x)\sin(y/z) + e^{xy}\cos(y/z)\cdot(1/z)}$$

$$G_z = \frac{\partial}{\partial z}\left(e^{xy} \sin(y/z)\right) = \boxed{e^{xy}\cos(y/z)\cdot(-y/z^2)}$$

3. If $v = x^2 \sin y + ye^{xy}$ where $x = s + 2t$ and $y = st$, use the chain rule to find $\frac{\partial v}{\partial t}$ and $\frac{\partial v}{\partial s}$ when $s = 0$ and $t = 1$.

$s=0, t=1 \Rightarrow x=2, y=0$ $\quad \frac{\partial x}{\partial t}\Big|_{(0,1)} = 2 \quad \frac{\partial x}{\partial s}\Big|_{(0,1)} = 1 \quad \frac{\partial y}{\partial t}\Big|_{(0,1)} = 0 \quad \frac{\partial y}{\partial s}\Big|_{(0,1)} = 1$

$$\frac{\partial v}{\partial \_} = 2x\frac{\partial x}{\partial \_}\sin(y) + x^2\cos y\frac{\partial y}{\partial \_} + \frac{\partial y}{\partial \_}e^{xy} + ye^{xy}\left(y\frac{\partial x}{\partial \_} + x\frac{\partial y}{\partial \_}\right)$$

$$\frac{\partial v}{\partial t}\Big|_{(0,1)} = 2(2)(2)\sin(0) + (2)^2\cos(0)(0) + (0)e^{2(0)} + (0)e^{(2)(0)}(0(2) + 2(0)) = \boxed{0}$$

$$\frac{\partial v}{\partial s}\Big|_{(0,1)} = 2(2)(1)\sin(0) + (2)^2\cos(0)(1) + (1)e^{2(0)} + (0)e^{(2)(0)}(0(1) + 2(1)) = \boxed{5}$$

4. Find the maximum rate of change of $f(x, y) = x^2y + \sqrt{x}$ at the point $(2, 1)$. In what direction does it occur?

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \left\langle 2xy + \frac{1}{2\sqrt{x}}, x^2 \right\rangle$$

Max rate at $(2,1)$ is $|\nabla f(2,1)| = \left|\left\langle 4 + \frac{\sqrt{2}}{4}, 4\right\rangle\right| = \boxed{\sqrt{\left(4+\frac{\sqrt{2}}{4}\right)^2 + 4^2}}$

in the direction of $\nabla f(2,1) = \boxed{\left\langle 4 + \frac{\sqrt{2}}{4}, 4 \right\rangle}$

5. Use Lagrange Multipliers to find the max/min values of $f(x, y) = \frac{1}{x} + \frac{1}{y}$ subject to the constraint $\frac{1}{x^2} + \frac{1}{y^2} = 1$. [Note I will only ask you for the set up of these problems]

$g(x,y) = x^{-2} + y^{-2}$

① $f_x = \lambda g_x$ ② $f_y = \lambda g_y$ ③ $g(x,y) = 1$

Set up:

$$\boxed{① \ -x^{-2} = \lambda(-2x^{-3}) \quad ② \ -y^{-2} = \lambda(-2y^{-3}) \quad ③ \ x^{-2} + y^{-2} = 1}$$

Solving: From ①, ② $\Rightarrow$ $x = 2\lambda$ and $y = 2\lambda$ into ③ gives $2(4\lambda^2)^{-1} = 1 \Rightarrow \lambda = \pm\frac{\sqrt{2}}{2}$

Thus $x = \pm\sqrt{2}$ and $y = \pm\sqrt{2}$ Points: $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$

$f(\sqrt{2}, \sqrt{2}) = \sqrt{2}$ $\quad f(-\sqrt{2}, -\sqrt{2}) = -\sqrt{2}$

max $\quad$ min

6. Find the absolute max and min values of f on the set D given:

$$f(x,y) = e^{-x^2-y^2}(x^2+2y^2) \quad \text{and} \quad D = \{(x,y) \mid x^2+y^2 \le 4\}$$

Critical points: $\vec{0} = \nabla f = e^{-x^2-y^2}\langle -2x(x^2+2y^2)+2x, -2y(x^2+2y^2)+4y \rangle$

$e^{-x^2-y^2} > 0$ so ① $2x(1-x^2-2y^2)=0$ ② $2y(2-x^2-2y^2)=0$

① $x=0$ or $x^2+2y^2=1 \Rightarrow y=0,\ x=\pm1$

② $y=0$ or $x^2+2y^2=2 \Rightarrow x=0,\ y=\pm1$

Points: $(0,0)$ $(0,\pm1)$, $(\pm1,0)$

Check Critical Points:

$f(0,0)=\underline{0}$ $\quad f(0,\pm1)=\frac{2}{e} \approx \underline{0.736}$ $\quad f(\pm1,0)=\frac{1}{e} \approx \underline{0.368}$

Check boundary: $x^2+y^2=4$ so $f(x,y)=e^{-4}(4+y^2)$ with $-2 \le y \le 2$

Max f when $y=\pm2$ $\quad f(x,\pm2)=e^{-4}(8) \approx \underline{0.398}$

min f when $y=0$ $\quad f(x,0)=e^{-4}(4) \approx \underline{0.0733}$

Absolute max value of f is $\frac{2}{e}$ at $(0,\pm1)$

Absolute min value of f is 0 at $(0,0)$