Review Practice: Chapter 13

1. Let $\mathbf{r}(t) = \left\langle \frac{\sin t}{t}, e^{\cos t}, \sqrt{4-t^2} \right\rangle$

(a) Find the domain of $\mathbf{r}$

$\underline{\frac{\sin t}{t} \Rightarrow t \neq 0}$; $\underline{e^{\cos t} \Rightarrow t \in \mathbb{R}}$; $\underline{\sqrt{4-t^2} \Rightarrow 2 \geq |t|}$ $\boxed{[2,0) \cup (0,2]}$

(b) Find $\lim_{t \to 0} \mathbf{r}(t)$

$$= \left\langle \underbrace{\lim_{t\to 0} \frac{\sin t}{t}}_{\text{L'Hopital}}, \underbrace{\lim_{t\to 0} e^{\cos t}, \lim_{t\to 0} \sqrt{4-t^2}}_{\text{both continuous}} \right\rangle = \left\langle \lim_{t\to 0} \frac{\cos t}{1}, e^{\cos(0)}, \sqrt{4-0^2} \right\rangle$$

$$= \boxed{\langle 1, e, 2 \rangle}$$

(c) Find $\mathbf{r}'(t)$

$$\vec{r}\,'(t) = \left\langle \underbrace{\frac{d}{dt}\left(\frac{\sin t}{t}\right)}_{\text{Quotient Rule}}, \underbrace{\frac{d}{dt}\left(e^{\cos t}\right), \frac{d}{dt}\left(\sqrt{4-t^2}\right)}_{\text{Chain Rule}} \right\rangle = \boxed{\left\langle \frac{t\cos t - \sin t}{t^2}, -\sin t\, e^{\cos t}, \frac{-t}{\sqrt{4-t^2}} \right\rangle}$$

2. Find a vector function that represents the curve on intersection of

$$x^2 + y^2 + z = 4 \quad \text{and} \quad x^2 + y^2 = 9$$

$x^2 + y^2 = 9$

$x = 3\cos t$

$y = 3\sin t$

$\underbrace{x^2 + y^2}+ z = 4$

$9 + z = 4$

$z = -5$

$\boxed{\vec{r}(t) = \langle 3\cos t, 3\sin t, -5 \rangle}$

3. Reparametrize $\mathbf{r}(t) = \langle e^t, \cos e^t, \sin e^t \rangle$ with respect to arc length measured from $t = 0$ in the direction of increasing t.

$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t \sqrt{e^{2u} + e^{2u}\sin^2 e^u + e^{2u}\cos^2 e^u}\,du$$

$$= \int_0^t \sqrt{e^{2u}}\sqrt{1 + \sin^2 e^u + \cos^2 e^u}\,du = \int_0^t e^u \sqrt{2}\,du$$

$$= (e^t - 1)\sqrt{2} \Rightarrow t = \ln\left(\frac{s}{\sqrt{2}} + 1\right)$$

$$\boxed{\vec{r}(s) = \left\langle \frac{s}{\sqrt{2}} + 1, \cos\left(\frac{s}{\sqrt{2}} + 1\right), \sin\left(\frac{s}{\sqrt{2}} + 1\right) \right\rangle}$$

4. Find the curvature of the ellipse $x = 3\cos t$, $y = 4\sin t$ at the points $(3, 0)$ and $(0, 4)$.

$\vec{r}\,'(t) = \langle -3\sin t, 4\cos t, 0 \rangle$

$\vec{r}\,''(t) = \langle -3\cos t, -4\sin t, 0 \rangle$

$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = 12$

$|\vec{r}\,'(t)|^3 = (9\sin^2 t + 16\cos^2 t)^{3/2}$

$K(0) = \dfrac{12}{16^{3/2}} = \boxed{\dfrac{3}{16}}$

$K(\pi/2) = \dfrac{12}{9^{3/2}} = \boxed{\dfrac{4}{9}}$

5. A particle starts at the origin with initial velocity $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and its acceleration is $\mathbf{a}(t) = \langle 6t, 12t^2, -6t \rangle$. Find its position function and its speed funtion.

$\vec{v}(t) = \int \vec{a}(t)\, dt = \langle 3t^2, 4t^3, -3t^2 \rangle + \langle 1, -1, 3 \rangle$

$\vec{r}(t) = \int \vec{v}(t)\, dt = \boxed{\langle t^3 + t, t^4 - t, -t^3 - 3t \rangle + \vec{0}}$

$|\vec{v}(t)| = \boxed{\left((3t^2 + 1)^2 + (4t^3 - 1)^2 + (-3t^2 + 3)^2 \right)^{1/2}}$