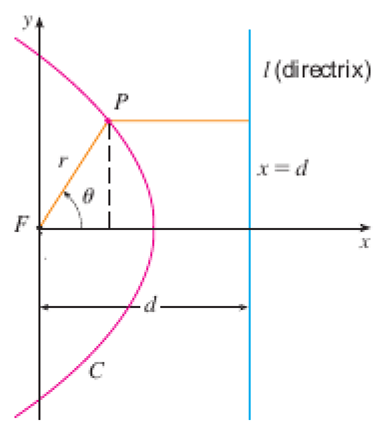
Kepler's Planetary Laws Project

1. Find and write down Kepler's three laws of planetary motion.
2. In Isaac Newton's book, *Principia Mathematic 1687*, he showed that Kepler's three laws of planetary motion where consequences of two of his own Laws. Research which of his two laws were used. State the two laws.

   Set up for proving Kepler's Law:
   Since the gravitational force of the sun on a planet is so much larger than forces exerted by other celestial bodies, we can safely ignore all bodies in the universe except the sun and the one planet revolving about it. Using a coordinate system with the sun at the origin, let $\mathbf{r}(t)$ be the position vector of the planet relative to the sun.
3. State Newton's Laws from part 2 in vector form, labeling constants.
4. For Kepler's first law, we begin by showing the planet moves in one plane. How can this be shown using vectors? (Hint: Draw a picture of a curve in the xy-plane, label vectors you know, and ask what would allow you to conclude the position of the planet is always in the plane.)
5. Prove what you stated in 4. (Hint: You will need to write the acceleration in terms of the position.)
6. How can you conclude that the orbit of the planet is in a plane?

   Next we prove that the planet's orbit is an ellipse. It is convenient to choose a coordinate axes so that the planet is moving in the xy-plane. In order to show the path is an ellipse you need to know the standard form of an ellipse and the polar form involving the eccentricity constant, e, a fixed positive number.
7. What is the standard form of an ellipse with axis length $2a$ along the x-axis and axis length $2b$ along the y-axis?
8. Let F be the focus, l the directix and e the eccentricity. Show that an ellise is the set of all points with $\dfrac{|PF|}{|Pl|} = e$ if $e < 1$. (Hint: You will need to rewrite the equation so that r and θ are eliminated then complete the square to put into the standard form.)

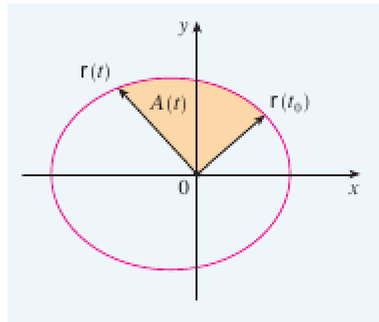


9. Now we see the form that we need to get $r = |\mathbf{r}|$ into to prove it's an ellipse. Show that $\mathbf{a} \times (\mathbf{r} \times \mathbf{v}) = GM\mathbf{u}'$ where $\mathbf{u}$ is a unit vector in the direction of $\mathbf{r}$.
10. Let $\mathbf{h} = \mathbf{r} \times \mathbf{v}$. Show $\mathbf{v} \times \mathbf{h} = GM\mathbf{u} + \mathbf{c}$ with $\mathbf{c}$ in the xy-plane. (Hint: Differential $\mathbf{v} \times \mathbf{h}$ and then integrate.)

11. Show $\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = h^2$ and that $\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = GMr + rc\cos\theta$ where $h = |\mathbf{h}|$ and $c = |\mathbf{c}|$.

12. Conclude you have the polar equation of an ellipse. This proves Kelper's first law.

13. Next to prove Kepler's second law let $A(t)$ be the area swept out by the position $\mathbf{r}(t)$ over $[t_0, t]$as shown in the figure. Use polar coordinates to express $\mathbf{r}(t)$ in terms of $r = |\mathbf{r}|$ and θ.



14. What needs to be shown to conclude Kepler's second law?

15. Show that $h = r^2 \dfrac{d\theta}{dt}$. Where $h = |\mathbf{h}| = |\mathbf{r} \times \mathbf{v}|$.

16. Show that:
$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$
.

17. Conlude Kepler's second law.

    Lastly, to prove Kepler's third law let T be the period of the planet about the sun and let the major and minor axes lengths be $2a$ and $2b$ respectively.

18. Show $T = \dfrac{2\pi ab}{h}$.

19. Show $\dfrac{h^2}{GM} = \dfrac{b^2}{a}$. (Hint: You will need to look at your work in 8 and 12.)

20. Show $T^2 = \dfrac{4\pi^2}{GM} a^3$ and conclude Kepler's third law.