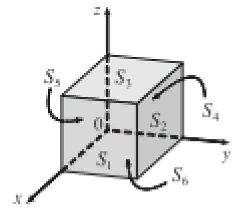
Section 16.9 - 1, 3, 6, 9, 12, 13, 23



1. div $\mathbf{F} = 3 + x + 2x = 3 + 3x$, so

$\iiint_E \operatorname{div} \mathbf{F}\, dV = \int_0^1 \int_0^1 \int_0^1 (3x + 3)\, dx\, dy\, dz = \frac{9}{2}$ (notice the triple integral is three times the volume of the cube plus three times $\overline{x}$).

To compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, on

S_1: $\mathbf{n} = \mathbf{i}$, $\mathbf{F} = 3\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$, and $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} 3\, dS = 3$;

S_2: $\mathbf{F} = 3x\,\mathbf{i} + x\,\mathbf{j} + 2xz\,\mathbf{k}$, $\mathbf{n} = \mathbf{j}$ and $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} x\, dS = \frac{1}{2}$;

S_3: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2x\,\mathbf{k}$, $\mathbf{n} = \mathbf{k}$ and $\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_3} 2x\, dS = 1$;

S_4: $\mathbf{F} = \mathbf{0}$, $\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = 0$; S_5: $\mathbf{F} = 3x\,\mathbf{i} + 2x\,\mathbf{k}$, $\mathbf{n} = -\mathbf{j}$ and $\iint_{S_5} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_5} 0\, dS = 0$;

S_6: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and $\iint_{S_6} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_6} 0\, dS = 0$. Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{9}{2}$.

3. div $\mathbf{F} = 0 + 1 + 0 = 1$, so $\iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E 1\, dV = V(E) = \frac{4}{3}\pi \cdot 4^3 = \frac{256}{3}\pi$. S is a sphere of radius 4 centered at the origin which can be parametrized by $\mathbf{r}(\phi, \theta) = \langle 4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi\rangle$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$ (similar to Example 16.6.10). Then

$$\begin{aligned}\mathbf{r}_\phi \times \mathbf{r}_\theta &= \langle 4\cos\phi\cos\theta, 4\cos\phi\sin\theta, -4\sin\phi\rangle \times \langle -4\sin\phi\sin\theta, 4\sin\phi\cos\theta, 0\rangle \\ &= \left\langle 16\sin^2\phi\cos\theta, 16\sin^2\phi\sin\theta, 16\cos\phi\sin\phi\right\rangle\end{aligned}$$

and $\mathbf{F}(\mathbf{r}(\phi, \theta)) = \langle 4\cos\phi, 4\sin\phi\sin\theta, 4\sin\phi\cos\theta\rangle$. Thus

$\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 64\cos\phi\sin^2\phi\cos\theta + 64\sin^3\phi\sin^2\theta + 64\cos\phi\sin^2\phi\cos\theta = 128\cos\phi\sin^2\phi\cos\theta + 64\sin^3\phi\sin^2\theta$

and

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, dA = \int_0^{2\pi} \int_0^{\pi} (128\cos\phi\sin^2\phi\cos\theta + 64\sin^3\phi\sin^2\theta)\, d\phi\, d\theta \\ &= \int_0^{2\pi} \left[\tfrac{128}{3}\sin^3\phi\cos\theta + 64\left(-\tfrac{1}{3}(2 + \sin^2\phi)\cos\phi\right)\sin^2\theta\right]_{\phi=0}^{\phi=\pi} d\theta \\ &= \int_0^{2\pi} \tfrac{256}{3}\sin^2\theta\, d\theta = \tfrac{256}{3}\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{256}{3}\pi\end{aligned}$$

6. div $\mathbf{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2) = 2xyz + 2xyz + 2xyz = 6xyz$, so by the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F}\, dV = \int_0^a \int_0^b \int_0^c 6xyz\, dz\, dy\, dx = 6\int_0^a x\, dx \int_0^b y\, dy \int_0^c z\, dz \\ &= 6\left[\tfrac{1}{2}x^2\right]_0^a \left[\tfrac{1}{2}y^2\right]_0^b \left[\tfrac{1}{2}z^2\right]_0^c = 6\left(\tfrac{1}{2}a^2\right)\left(\tfrac{1}{2}b^2\right)\left(\tfrac{1}{2}c^2\right) = \tfrac{3}{4}a^2b^2c^2\end{aligned}$$

9. div $\mathbf{F} = 2x\sin y - x\sin y - x\sin y = 0$, so by the Divergence Theorem, $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 0\, dV = 0$.

12. div $\mathbf{F} = 4x^3 + 4xy^2$ so

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E 4x(x^2 + y^2)\, dV = \int_0^{2\pi} \int_0^1 \int_0^{r\cos\theta + 2} (4r^3\cos\theta)\, r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 (4r^5\cos^2\theta + 8r^4\cos\theta)\, dr\, d\theta = \int_0^{2\pi} \left(\tfrac{2}{3}\cos^2\theta + \tfrac{8}{5}\cos\theta\right) d\theta = \tfrac{2}{3}\pi\end{aligned}$$

13. $\mathbf{F}(x,y,z) = x\sqrt{x^2+y^2+z^2}\,\mathbf{i} + y\sqrt{x^2+y^2+z^2}\,\mathbf{j} + z\sqrt{x^2+y^2+z^2}\,\mathbf{k}$, so

$$\begin{aligned}\operatorname{div}\mathbf{F} &= x\cdot\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x) + (x^2+y^2+z^2)^{1/2} + y\cdot\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y) + (x^2+y^2+z^2)^{1/2} \\ &\quad + z\cdot\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z) + (x^2+y^2+z^2)^{1/2} \\ &= (x^2+y^2+z^2)^{-1/2}\left[x^2+(x^2+y^2+z^2)+y^2+(x^2+y^2+z^2)+z^2+(x^2+y^2+z^2)\right] \\ &= \frac{4(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} = 4\sqrt{x^2+y^2+z^2}.\end{aligned}$$

Then

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 4\sqrt{x^2+y^2+z^2}\,dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 4\sqrt{\rho^2}\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \int_0^{\pi/2}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^1 4\rho^3\,d\rho = [-\cos\phi]_0^{\pi/2}\,[\theta]_0^{2\pi}\,\left[\rho^4\right]_0^1 = (1)\,(2\pi)\,(1) = 2\pi\end{aligned}$$

23. Since $\dfrac{\mathbf{x}}{|\mathbf{x}|^3} = \dfrac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$ and $\dfrac{\partial}{\partial x}\left(\dfrac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \dfrac{(x^2+y^2+z^2)-3x^2}{(x^2+y^2+z^2)^{5/2}}$ with similar expressions for $\dfrac{\partial}{\partial y}\left(\dfrac{y}{(x^2+y^2+z^2)^{3/2}}\right)$ and $\dfrac{\partial}{\partial z}\left(\dfrac{z}{(x^2+y^2+z^2)^{3/2}}\right)$, we have

$\operatorname{div}\left(\dfrac{\mathbf{x}}{|\mathbf{x}|^3}\right) = \dfrac{3(x^2+y^2+z^2)-3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$, except at $(0,0,0)$ where it is undefined.