Section 16.8 - 1, 2, 5, 7, 10, 14, 17

1. Both H and P are oriented piecewise-smooth surfaces that are bounded by the simple, closed, smooth curve $x^2 + y^2 = 4$, $z = 0$ (which we can take to be oriented positively for both surfaces). Then H and P satisfy the hypotheses of Stokes' Theorem, so by (3) we know $\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ (where C is the boundary curve).

2. The boundary curve C is the circle $x^2 + y^2 = 9$, $z = 0$ oriented in the counterclockwise direction when viewed from above. A vector equation of C is $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$, so $\mathbf{r}'(t) = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j}$ and $\mathbf{F}(\mathbf{r}(t)) = 2(3\sin t)(\cos 0)\,\mathbf{i} + e^{3\cos t}(\sin 0)\,\mathbf{j} + (3\cos t)e^{3\sin t}\,\mathbf{k} = 6\sin t\,\mathbf{i} + (3\cos t)e^{3\sin t}\,\mathbf{k}$. Then, by Stokes' Theorem, $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} (-18\sin^2 t + 0 + 0)\,dt = -18\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{2\pi} = -18\pi$.

5. C is the square in the plane $z = -1$. Rather than evaluating a line integral around C we can use Equation 3: $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ where S_1 is the original cube without the bottom and S_2 is the bottom face of the cube. $\operatorname{curl} \mathbf{F} = x^2 z\,\mathbf{i} + (xy - 2xyz)\,\mathbf{j} + (y - xz)\,\mathbf{k}$. For S_2, we choose $\mathbf{n} = \mathbf{k}$ so that C has the same orientation for both surfaces. Then $\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = y - xz = x + y$ on S_2, where $z = -1$. Thus $\iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1} \int_{-1}^{1} (x + y)\,dx\,dy = 0$ so $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

7. $\operatorname{curl} \mathbf{F} = -2z\,\mathbf{i} - 2x\,\mathbf{j} - 2y\,\mathbf{k}$ and we take the surface S to be the planar region enclosed by C, so S is the portion of the plane $x + y + z = 1$ over $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$. Since C is oriented counterclockwise, we orient S upward. Using Equation 16.7.10, we have $z = g(x, y) = 1 - x - y$, $P = -2z$, $Q = -2x$, $R = -2y$, and

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D [-(-2z)(-1) - (-2x)(-1) + (-2y)]\,dA \\ &= \int_0^1 \int_0^{1-x} (-2)\,dy\,dx = -2\int_0^1 (1 - x)\,dx = -1\end{aligned}$$

10. The curve of intersection is an ellipse in the plane $z = 5 - x$. $\operatorname{curl} \mathbf{F} = \mathbf{i} - x\,\mathbf{k}$ and we take the surface S to be the planar region enclosed by C with upward orientation, so

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2+y^2 \le 9} [-1\,(-1) - 0 + (-x)]\,dA = \int_0^{2\pi} \int_0^3 (1 - r\cos\theta)\;r\,dr\,d\theta \\ &= \int_0^{2\pi} \int_0^3 \left(r - r^2\cos\theta\right) dr\,d\theta = \int_0^{2\pi} \left(\tfrac{9}{2} - 9\cos\theta\right) d\theta = \left[\tfrac{9}{2}\theta - 9\sin\theta\right]_0^{2\pi} = 9\pi\end{aligned}$$

14. The paraboloid intersects the plane $z = 1$ when $1 = 5 - x^2 - y^2 \quad \Leftrightarrow \quad x^2 + y^2 = 4$, so the boundary curve C is the circle $x^2 + y^2 = 4$, $z = 1$ oriented in the counterclockwise direction as viewed from above. We can parametrize C by $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$, and then $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$. Thus $\mathbf{F}(\mathbf{r}(t)) = -4\sin t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 6\cos t\,\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 8\sin^2 t + 4\sin t\cos t$, and

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (8\sin^2 t + 4\sin t\cos t)\,dt = 8\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right) + 2\sin^2 t\Big]_0^{2\pi} = 8\pi$$

Now $\operatorname{curl}\mathbf{F} = (-3 - 2y)\,\mathbf{j} + 2z\,\mathbf{k}$, and the projection D of S on the xy-plane is the disk $x^2 + y^2 \le 4$, so by Equation 16.7.10 with $z = g(x, y) = 5 - x^2 - y^2$ we have

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} &= \iint_D [-0 - (-3 - 2y)(-2y) + 2z]\,dA = \iint_D [-6y - 4y^2 + 2(5 - x^2 - y^2)]\,dA \\ &= \int_0^{2\pi}\int_0^2 \left[-6r\sin\theta - 4r^2\sin^2\theta + 2(5 - r^2)\right] r\,dr\,d\theta = \int_0^{2\pi} \left[-2r^3\sin\theta - r^4\sin^2\theta + 5r^2 - \tfrac{1}{2}r^4\right]_{r=0}^{r=2} d\theta \\ &= \int_0^{2\pi} \left(-16\sin\theta - 16\sin^2\theta + 20 - 8\right) d\theta = 16\cos\theta - 16\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right) + 12\theta\Big]_0^{2\pi} = 8\pi\end{aligned}$$

17. It is easier to use Stokes' Theorem than to compute the work directly. Let S be the planar region enclosed by the path of the particle, so S is the portion of the plane $z = \frac{1}{2}y$ for $0 \le x \le 1$, $0 \le y \le 2$, with upward orientation.

$\operatorname{curl}\mathbf{F} = 8y\,\mathbf{i} + 2z\,\mathbf{j} + 2y\,\mathbf{k}$ and

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D \left[-8y\,(0) - 2z\left(\tfrac{1}{2}\right) + 2y\right] dA = \int_0^1\int_0^2 \left(2y - \tfrac{1}{2}y\right) dy\,dx \\ &= \int_0^1\int_0^2 \tfrac{3}{2}y\,dy\,dx = \int_0^1 \left[\tfrac{3}{4}y^2\right]_{y=0}^{y=2} dx = \int_0^1 3\,dx = 3\end{aligned}$$