Section 16.7 - 2, 3, 5, 9, 11, 19, 21, 24, 25

2. Each quarter-cylinder has surface area $\frac{1}{4}[2\pi(1)(2)] = \pi$, and the top and bottom disks have surface area $\pi(1)^2 = \pi$. We can take $(0,0,1)$ as a sample point in the top disk, $(0,0,-1)$ in the bottom disk, and $(\pm 1,0,0)$, $(0,\pm 1,0)$ in the four quarter-cylinders. Then $\iint_S f(x,y,z)\,dS$ can be approximated by the Riemann sum

$$f(1,0,0)(\pi) + f(-1,0,0)(\pi) + f(0,1,0)\,(\pi) + f(0,-1,0)(\pi) + f(0,0,1)(\pi) + f(0,0,-1)(\pi)$$
$$= (2+2+3+3+4+4)\pi = 18\pi \approx 56.5.$$

3. We can use the xz- and yz-planes to divide H into four patches of equal size, each with surface area equal to $\frac{1}{8}$ the surface area of a sphere with radius $\sqrt{50}$, so $\Delta S = \frac{1}{8}(4)\pi\left(\sqrt{50}\right)^2 = 25\pi$. Then $(\pm 3, \pm 4, 5)$ are sample points in the four patches, and using a Riemann sum as in Definition 1, we have

$$\iint_H f(x,y,z)\,dS \approx f(3,4,5)\,\Delta S + f(3,-4,5)\,\Delta S + f(-3,4,5)\,\Delta S + f(-3,-4,5)\,\Delta S$$
$$= (7+8+9+12)(25\pi) = 900\pi \approx 2827$$

5. $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + (u-v)\,\mathbf{j} + (1+2u+v)\,\mathbf{k}$, $0 \le u \le 2$, $0 \le v \le 1$ and

$\mathbf{r}_u \times \mathbf{r}_v = (\mathbf{i}+\mathbf{j}+2\,\mathbf{k}) \times (\mathbf{i}-\mathbf{j}+\mathbf{k}) = 3\,\mathbf{i}+\mathbf{j}-2\,\mathbf{k} \quad\Rightarrow\quad |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{3^2+1^2+(-2)^2} = \sqrt{14}$. Then by Formula 2,

$$\iint_S (x+y+z)\,dS = \iint_D (u+v+u-v+1+2u+v)\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA = \int_0^1\int_0^2 (4u+v+1)\cdot\sqrt{14}\,du\,dv$$
$$= \sqrt{14}\int_0^1 \left[2u^2+uv+u\right]_{u=0}^{u=2} dv = \sqrt{14}\int_0^1 (2v+10)\,dv = \sqrt{14}\left[v^2+10v\right]_0^1 = 11\sqrt{14}$$

9. $z = 1+2x+3y$ so $\dfrac{\partial z}{\partial x} = 2$ and $\dfrac{\partial z}{\partial y} = 3$. Then by Formula 4,

$$\iint_S x^2yz\,dS = \iint_D x^2yz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA = \int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{4+9+1}\,dy\,dx$$
$$= \sqrt{14}\int_0^3\int_0^2 (x^2y+2x^3y+3x^2y^2)\,dy\,dx = \sqrt{14}\int_0^3 \left[\tfrac{1}{2}x^2y^2 + x^3y^2 + x^2y^3\right]_{y=0}^{y=2} dx$$
$$= \sqrt{14}\int_0^3 (10x^2+4x^3)\,dx = \sqrt{14}\left[\tfrac{10}{3}x^3 + x^4\right]_0^3 = 171\sqrt{14}$$

11. An equation of the plane through the points $(1,0,0)$, $(0,-2,0)$, and $(0,0,4)$ is $4x-2y+z=4$, so S is the region in the plane $z = 4-4x+2y$ over $D = \{(x,y) \mid 0 \le x \le 1, 2x-2 \le y \le 0\}$. Thus by Formula 4,

$$\iint_S x\,dS = \iint_D x\sqrt{(-4)^2+(2)^2+1}\,dA = \sqrt{21}\int_0^1\int_{2x-2}^0 x\,dy\,dx = \sqrt{21}\int_0^1 [xy]_{y=2x-2}^{y=0}\,dx$$
$$= \sqrt{21}\int_0^1 (-2x^2+2x)\,dx = \sqrt{21}\left[-\tfrac{2}{3}x^3+x^2\right]_0^1 = \sqrt{21}\left(-\tfrac{2}{3}+1\right) = \tfrac{\sqrt{21}}{3}$$

19. S is given by $\mathbf{r}(u,v) = u\,\mathbf{i} + \cos v\,\mathbf{j} + \sin v\,\mathbf{k}$, $0 \le u \le 3$, $0 \le v \le \pi/2$. Then

$\mathbf{r}_u \times \mathbf{r}_v = \mathbf{i} \times (-\sin v\,\mathbf{j} + \cos v\,\mathbf{k}) = -\cos v\,\mathbf{j} - \sin v\,\mathbf{k}$ and $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\cos^2 v + \sin^2 v} = 1$, so

$$\iint_S (z+x^2y)\,dS = \int_0^{\pi/2}\int_0^3 (\sin v + u^2\cos v)(1)\,du\,dv = \int_0^{\pi/2} (3\sin v + 9\cos v)\,dv$$
$$= \left[-3\cos v + 9\sin v\right]_0^{\pi/2} = 0+9+3-0 = 12$$

21. From Exercise 5, $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + (u-v)\,\mathbf{j} + (1+2u+v)\,\mathbf{k}$, $0 \le u \le 2$, $0 \le v \le 1$, and $\mathbf{r}_u \times \mathbf{r}_v = 3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$.

Then

$$\begin{aligned}\mathbf{F}(\mathbf{r}(u,v)) &= (1+2u+v)e^{(u+v)(u-v)}\,\mathbf{i} - 3(1+2u+v)e^{(u+v)(u-v)}\,\mathbf{j} + (u+v)(u-v)\,\mathbf{k}\\ &= (1+2u+v)e^{u^2-v^2}\,\mathbf{i} - 3(1+2u+v)e^{u^2-v^2}\,\mathbf{j} + (u^2-v^2)\,\mathbf{k}\end{aligned}$$

Because the z-component of $\mathbf{r}_u \times \mathbf{r}_v$ is negative we use $-(\mathbf{r}_u \times \mathbf{r}_v)$ in Formula 9 for the upward orientation:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\cdot(-(\mathbf{r}_u \times \mathbf{r}_v))\,dA = \int_0^1\int_0^2 \left[-3(1+2u+v)e^{u^2-v^2} + 3(1+2u+v)e^{u^2-v^2} + 2(u^2-v^2)\right] du\,dv\\ &= \int_0^1\int_0^2 2(u^2-v^2)\,du\,dv = 2\int_0^1 \left[\tfrac{1}{3}u^3 - uv^2\right]_{u=0}^{u=2} dv = 2\int_0^1 \left(\tfrac{8}{3} - 2v^2\right) dv\\ &= 2\left[\tfrac{8}{3}v - \tfrac{2}{3}v^3\right]_0^1 = 2\left(\tfrac{8}{3} - \tfrac{2}{3}\right) = 4\end{aligned}$$

24. $\mathbf{F}(x,y,z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^3\,\mathbf{k}$, $z = g(x,y) = \sqrt{x^2+y^2}$, and D is the annular region $\{(x,y) \mid 1 \le x^2+y^2 \le 9\}$. Since S has downward orientation, we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= -\iint_D \left[-(-x)\left(\frac{x}{\sqrt{x^2+y^2}}\right) - (-y)\left(\frac{y}{\sqrt{x^2+y^2}}\right) + z^3\right] dA\\ &= -\iint_D \left[\frac{x^2+y^2}{\sqrt{x^2+y^2}} + \left(\sqrt{x^2+y^2}\right)^3\right] dA = -\int_0^{2\pi}\int_1^3 \left(\frac{r^2}{r} + r^3\right) r\,dr\,d\theta\\ &= -\int_0^{2\pi} d\theta \int_1^3 (r^2 + r^4)\,dr = -\left[\theta\right]_0^{2\pi} \left[\tfrac{1}{3}r^3 + \tfrac{1}{5}r^5\right]_1^3\\ &= -2\pi\left(9 + \tfrac{243}{5} - \tfrac{1}{3} - \tfrac{1}{5}\right) = -\tfrac{1712}{15}\pi\end{aligned}$$

25. $\mathbf{F}(x,y,z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$, $z = g(x,y) = \sqrt{4-x^2-y^2}$ and D is the quarter disk $\left\{(x,y) \,\middle|\, 0 \le x \le 2, 0 \le y \le \sqrt{4-x^2}\right\}$. S has downward orientation, so by Formula 10,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= -\iint_D \left[-x\cdot\tfrac{1}{2}(4-x^2-y^2)^{-1/2}(-2x) - (-z)\cdot\tfrac{1}{2}(4-x^2-y^2)^{-1/2}(-2y) + y\right] dA\\ &= -\iint_D \left(\frac{x^2}{\sqrt{4-x^2-y^2}} - \sqrt{4-x^2-y^2}\cdot\frac{y}{\sqrt{4-x^2-y^2}} + y\right) dA\\ &= -\iint_D x^2(4-(x^2+y^2))^{-1/2}\,dA = -\int_0^{\pi/2}\int_0^2 (r\cos\theta)^2(4-r^2)^{-1/2}\,r\,dr\,d\theta\\ &= -\int_0^{\pi/2}\cos^2\theta\,d\theta \int_0^2 r^3(4-r^2)^{-1/2}\,dr \qquad \left[\text{let } u = 4-r^2 \;\Rightarrow\; r^2 = 4-u \text{ and } -\tfrac{1}{2}\,du = r\,dr\right]\\ &= -\int_0^{\pi/2}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right) d\theta \int_4^0 -\tfrac{1}{2}(4-u)(u)^{-1/2}\,du\\ &= -\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2} \left(-\tfrac{1}{2}\right)\left[8\sqrt{u} - \tfrac{2}{3}u^{3/2}\right]_4^0 = -\tfrac{\pi}{4}\left(-\tfrac{1}{2}\right)\left(-16 + \tfrac{16}{3}\right) = -\tfrac{4}{3}\pi\end{aligned}$$