Section 16.5 - 5, 7, 9-11, 12, 18, 19, 21

5. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}}\left[(-yz+yz)\,\mathbf{i} - (-xz+xz)\,\mathbf{j} + (-xy+xy)\,\mathbf{k}\right] = \mathbf{0}$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{x}{\sqrt{x^2+y^2+z^2}}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{y}{\sqrt{x^2+y^2+z^2}}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)$

$$= \frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2+z^2-y^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2+z^2-z^2}{(x^2+y^2+z^2)^{3/2}} = \frac{2x^2+2y^2+2z^2}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{\sqrt{x^2+y^2+z^2}}$$

7. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x\sin y & e^y\sin z & e^z\sin x \end{vmatrix} = (0 - e^y\cos z)\,\mathbf{i} - (e^z\cos x - 0)\,\mathbf{j} + (0 - e^x\cos y)\,\mathbf{k}$

$$= \langle -e^y\cos z, -e^z\cos x, -e^x\cos y\rangle$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(e^x\sin y) + \dfrac{\partial}{\partial y}(e^y\sin z) + \dfrac{\partial}{\partial z}(e^z\sin x) = e^x\sin y + e^y\sin z + e^z\sin x$

9. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, the x-component of each vector of $\mathbf{F}$ is 0, so $P = 0$, hence $\dfrac{\partial P}{\partial x} = \dfrac{\partial P}{\partial y} = \dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = 0$. Q decreases as y increases, so $\dfrac{\partial Q}{\partial y} < 0$, but Q doesn't change in the x- or z-directions, so $\dfrac{\partial Q}{\partial x} = \dfrac{\partial Q}{\partial z} = 0$.

(a) $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = 0 + \dfrac{\partial Q}{\partial y} + 0 < 0$

(b) $\operatorname{curl}\mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$

10. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, P and Q don't vary in the z-direction, so $\dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = \dfrac{\partial P}{\partial z} = \dfrac{\partial Q}{\partial z} = 0$. As x increases, the x-component of each vector of $\mathbf{F}$ increases while the y-component remains constant, so $\dfrac{\partial P}{\partial x} > 0$ and $\dfrac{\partial Q}{\partial x} = 0$. Similarly, as y increases, the y-component of each vector increases while the x-component remains constant, so $\dfrac{\partial Q}{\partial y} > 0$ and $\dfrac{\partial P}{\partial y} = 0$.

(a) $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + 0 > 0$

(b) $\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$

11. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, the y-component of each vector of $\mathbf{F}$ is 0, so $Q = 0$, hence $\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} = \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial x} = \frac{\partial R}{\partial y} = \frac{\partial R}{\partial z} = 0$. P increases as y increases, so $\frac{\partial P}{\partial y} > 0$, but P doesn't change in the x- or z-directions, so $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial z} = 0$.

(a) $\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 + 0 + 0 = 0$

(b) $\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + \left(0 - \frac{\partial P}{\partial y}\right)\mathbf{k} = -\frac{\partial P}{\partial y}\mathbf{k}$

Since $\frac{\partial P}{\partial y} > 0$, $-\frac{\partial P}{\partial y}\mathbf{k}$ is a vector pointing in the negative z-direction.

12. (a) $\operatorname{curl} f = \nabla \times f$ is meaningless because f is a scalar field.

(b) $\operatorname{grad} f$ is a vector field.

(c) $\operatorname{div}\mathbf{F}$ is a scalar field.

(d) $\operatorname{curl}(\operatorname{grad} f)$ is a vector field.

(e) $\operatorname{grad}\mathbf{F}$ is meaningless because $\mathbf{F}$ is not a scalar field.

(f) $\operatorname{grad}(\operatorname{div}\mathbf{F})$ is a vector field.

(g) $\operatorname{div}(\operatorname{grad} f)$ is a scalar field.

(h) $\operatorname{grad}(\operatorname{div} f)$ is meaningless because f is a scalar field.

(i) $\operatorname{curl}(\operatorname{curl}\mathbf{F})$ is a vector field.

(j) $\operatorname{div}(\operatorname{div}\mathbf{F})$ is meaningless because $\operatorname{div}\mathbf{F}$ is a scalar field.

(k) $(\operatorname{grad} f) \times (\operatorname{div}\mathbf{F})$ is meaningless because $\operatorname{div}\mathbf{F}$ is a scalar field.

(l) $\operatorname{div}(\operatorname{curl}(\operatorname{grad} f))$ is a scalar field.

18. $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x \sin yz & ze^x \cos yz & ye^x \cos yz \end{vmatrix}$

$$= [-yze^x \sin yz + e^x \cos yz - (-yze^x \sin yz + e^x \cos yz)]\,\mathbf{i} - (ye^x \cos yz - ye^x \cos yz)\,\mathbf{j} + (ze^x \cos yz - ze^x \cos yz)\,\mathbf{k} = \mathbf{0}$$

$\mathbf{F}$ is defined on all of $\mathbb{R}^3$, and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = e^x \sin yz$ implies $f(x,y,z) = e^x \sin yz + g(y,z) \Rightarrow$ $f_y(x,y,z) = ze^x \cos yz + g_y(y,z)$. But $f_y(x,y,z) = ze^x \cos yz$, so $g(y,z) = h(z)$ and $f(x,y,z) = e^x \sin yz + h(z)$. Thus $f_z(x,y,z) = ye^x \cos yz + h'(z)$ but $f_z(x,y,z) = ye^x \cos yz$ so $h(z) = K$ and a potential function for $\mathbf{F}$ is $f(x,y,z) = e^x \sin yz + K$.

19. No. Assume there is such a $\mathbf{G}$. Then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = \dfrac{\partial}{\partial x}(x\sin y) + \dfrac{\partial}{\partial y}(\cos y) + \dfrac{\partial}{\partial z}(z - xy) = \sin y - \sin y + 1 \neq 0$, which contradicts Theorem 11.

21. $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$. Hence $\mathbf{F} = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$ is irrotational.