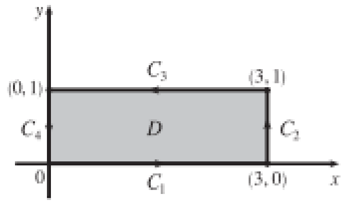
2. (a)



C_1: $x = t \;\Rightarrow\; dx = dt,\; y = 0 \;\Rightarrow\; dy = 0\,dt,\; 0 \le t \le 3.$

C_2: $x = 3 \;\Rightarrow\; dx = 0\,dt,\; y = t \;\Rightarrow\; dy = dt,\; 0 \le t \le 1.$

C_3: $x = 3 - t \;\Rightarrow\; dx = -dt,\; y = 1 \;\Rightarrow\; dy = 0\,dt,\; 0 \le t \le 3.$

C_4: $x = 0 \;\Rightarrow\; dx = 0\,dt,\; y = 1 - t \;\Rightarrow\; dy = -dt,\; 0 \le t \le 1$

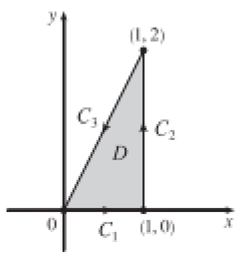
Thus

$$\oint_C xy\,dx + x^2\,dy = \oint_{C_1 + C_2 + C_3 + C_4} xy\,dx + x^2\,dy = \int_0^3 0\,dt + \int_0^1 9\,dt + \int_0^3 (3-t)(-1)\,dt + \int_0^1 0\,dt$$

$$= \big[9t\big]_0^1 + \left[\tfrac{1}{2}t^2 - 3t\right]_0^3 = 9 + \tfrac{9}{2} - 9 = \tfrac{9}{2}$$

(b) $\oint_C xy\,dx + x^2\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^3 \int_0^1 (2x - x)\,dy\,dx = \int_0^3 x\,dx \int_0^1 dy = \left[\tfrac{1}{2}x^2\right]_0^3 \cdot 1 = \tfrac{9}{2}$

3. (a)



C_1: $x = t \;\Rightarrow\; dx = dt,\; y = 0 \;\Rightarrow\; dy = 0\,dt,\; 0 \le t \le 1.$

C_2: $x = 1 \;\Rightarrow\; dx = 0\,dt,\; y = t \;\Rightarrow\; dy = dt,\; 0 \le t \le 2.$

C_3: $x = 1 - t \;\Rightarrow\; dx = -dt,\; y = 2 - 2t \;\Rightarrow\; dy = -2\,dt,\; 0 \le t \le 1.$

Thus

$$\oint_C xy\,dx + x^2y^3\,dy = \oint_{C_1 + C_2 + C_3} xy\,dx + x^2y^3\,dy$$

$$= \int_0^1 0\,dt + \int_0^2 t^3\,dt + \int_0^1 \left[-(1-t)(2-2t) - 2(1-t)^2(2-2t)^3\right] dt$$

$$= 0 + \left[\tfrac{1}{4}t^4\right]_0^2 + \left[\tfrac{2}{3}(1-t)^3 + \tfrac{8}{3}(1-t)^6\right]_0^1 = 4 - \tfrac{10}{3} = \tfrac{2}{3}$$

(b) $\oint_C xy\,dx + x^2y^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$

$$= \int_0^1 \left[\tfrac{1}{2}xy^4 - xy\right]_{y=0}^{y=2x} dx = \int_0^1 (8x^5 - 2x^2)\,dx = \tfrac{4}{3} - \tfrac{2}{3} = \tfrac{2}{3}$$

7. $\int_C \left(y + e^{\sqrt{x}}\right) dx + (2x + \cos y^2)\,dy = \iint_D \left[\frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}\left(y + e^{\sqrt{x}}\right)\right] dA$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2 - 1)\,dx\,dy = \int_0^1 (y^{1/2} - y^2)\,dy = \tfrac{1}{3}$$

10. $\int_C (1 - y^3)\,dx + (x^3 + e^{y^2})\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^3 + e^{y^2}) - \frac{\partial}{\partial y}(1 - y^3)\right] dA = \iint_D (3x^2 + 3y^2)\,dA$

$$= \int_0^{2\pi} \int_2^3 (3r^2)\,r\,dr\,d\theta = 3\int_0^{2\pi} d\theta \int_2^3 r^3\,dr$$

$$= 3\big[\theta\big]_0^{2\pi} \left[\tfrac{1}{4}r^4\right]_2^3 = 3(2\pi)\cdot\tfrac{1}{4}(81 - 16) = \tfrac{195}{2}\pi$$

13. $\mathbf{F}(x, y) = \langle y - \cos y, x \sin y \rangle$ and the region D enclosed by C is the disk with radius 2 centered at $(3, -4)$.

C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} (y - \cos y)\, dx + (x \sin y)\, dy = -\iint_D \left[\frac{\partial}{\partial x}(x \sin y) - \frac{\partial}{\partial y}(y - \cos y)\right] dA$$

$$= -\iint_D (\sin y - 1 - \sin y)\, dA = \iint_D dA = \text{area of } D = \pi(2)^2 = 4\pi$$

17. By Green's Theorem, $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x(x + y)\, dx + xy^2\, dy = \iint_D (y^2 - x)\, dA$ where C is the path described in the question and D is the triangle bounded by C. So

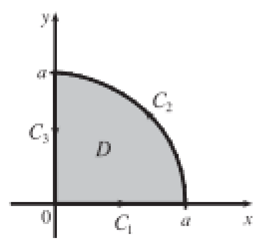
$$W = \int_0^1 \int_0^{1-x} (y^2 - x)\, dy\, dx = \int_0^1 \left[\tfrac{1}{3}y^3 - xy\right]_{y=0}^{y=1-x} dx = \int_0^1 \left(\tfrac{1}{3}(1 - x)^3 - x(1 - x)\right) dx$$

$$= \left[-\tfrac{1}{12}(1 - x)^4 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\right]_0^1 = \left(-\tfrac{1}{2} + \tfrac{1}{3}\right) - \left(-\tfrac{1}{12}\right) = -\tfrac{1}{12}$$

22. By Green's Theorem, $\frac{1}{2A}\oint_C x^2\, dy = \frac{1}{2A}\iint_D 2x\, dA = \frac{1}{A}\iint_D x\, dA = \overline{x}$ and

$-\frac{1}{2A}\oint_C y^2\, dx = -\frac{1}{2A}\iint_D (-2y)\, dA = \frac{1}{A}\iint_D y\, dA = \overline{y}$.

23. We orient the quarter-circular region as shown in the figure.



$A = \frac{1}{4}\pi a^2$ so $\overline{x} = \dfrac{1}{\pi a^2/2}\displaystyle\oint_C x^2\, dy$ and $\overline{y} = -\dfrac{1}{\pi a^2/2}\displaystyle\oint_C y^2 dx$.

Here $C = C_1 + C_2 + C_3$ where C_1: $x = t$, $y = 0$, $0 \le t \le a$;

C_2: $x = a\cos t$, $y = a \sin t$, $0 \le t \le \frac{\pi}{2}$; and

C_3: $x = 0, y = a - t, 0 \le t \le a$. Then

$$\oint_C x^2\, dy = \int_{C_1} x^2\, dy + \int_{C_2} x^2\, dy + \int_{C_3} x^2\, dy = \int_0^a 0\, dt + \int_0^{\pi/2} (a\cos t)^2 (a \cos t)\, dt + \int_0^a 0\, dt$$

$$= \int_0^{\pi/2} a^3 \cos^3 t\, dt = a^3 \int_0^{\pi/2} (1 - \sin^2 t) \cos t\, dt = a^3 \left[\sin t - \tfrac{1}{3}\sin^3 t\right]_0^{\pi/2} = \tfrac{2}{3}a^3$$

so $\overline{x} = \dfrac{1}{\pi a^2/2}\displaystyle\oint_C x^2\, dy = \dfrac{4a}{3\pi}$.

$$\oint_C y^2 dx = \int_{C_1} y^2\, dx + \int_{C_2} y^2\, dx + \int_{C_3} y^2\, dx = \int_0^a 0\, dt + \int_0^{\pi/2} (a \sin t)^2 (-a \sin t)\, dt + \int_0^a 0\, dt$$

$$= \int_0^{\pi/2} (-a^3 \sin^3 t)\, dt = -a^3 \int_0^{\pi/2} (1 - \cos^2 t) \sin t\, dt = -a^3 \left[\tfrac{1}{3}\cos^3 t - \cos t\right]_0^{\pi/2} = -\tfrac{2}{3}a^3,$$

so $\overline{y} = -\dfrac{1}{\pi a^2/2}\displaystyle\oint_C y^2 dx = \dfrac{4a}{3\pi}$. Thus $(\overline{x}, \overline{y}) = \left(\dfrac{4a}{3\pi}, \dfrac{4a}{3\pi}\right)$.