Section 16.3 - 1, 5, 9, 13, 17, 20, 21, 24, 26, 33

1. C appears to be a smooth curve, and since ∇f is continuous, we know f is differentiable. Then Theorem 2 says that the value of $\int_C \nabla f \cdot d\mathbf{r}$ is simply the difference of the values of f at the terminal and initial points of C. From the graph, this is $50 - 10 = 40$.

5. $\partial(e^x \cos y)/\partial y = -e^x \sin y$, $\partial(e^x \sin y)/\partial x = e^x \sin y$. Since these are not equal, $\mathbf{F}$ is not conservative.

9. $\partial(\ln y + 2xy^3)/\partial y = 1/y + 6xy^2 = \partial(3x^2y^2 + x/y)/\partial x$ and the domain of $\mathbf{F}$ is $\{(x, y) \mid y > 0\}$ which is open and simply connected. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = \ln y + 2xy^3$ implies $f(x, y) = x \ln y + x^2y^3 + g(y)$ and $f_y(x, y) = x/y + 3x^2y^2 + g'(y)$. But $f_y(x, y) = 3x^2y^2 + x/y$ so $g'(y) = 0 \Rightarrow g(y) = K$ and $f(x, y) = x \ln y + x^2y^3 + K$ is a potential function for $\mathbf{F}$.

13. (a) $f_x(x, y) = xy^2$ implies $f(x, y) = \frac{1}{2}x^2y^2 + g(y)$ and $f_y(x, y) = x^2y + g'(y)$. But $f_y(x, y) = x^2y$ so $g'(y) = 0 \Rightarrow g(y) = K$, a constant. We can take $K = 0$, so $f(x, y) = \frac{1}{2}x^2y^2$.

(b) The initial point of C is $\mathbf{r}(0) = (0, 1)$ and the terminal point is $\mathbf{r}(1) = (2, 1)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(2, 1) - f(0, 1) = 2 - 0 = 2$.

17. (a) $f_x(x, y, z) = yze^{xz}$ implies $f(x, y, z) = ye^{xz} + g(y, z)$ and so $f_y(x, y, z) = e^{xz} + g_y(y, z)$. But $f_y(x, y, z) = e^{xz}$ so $g_y(y, z) = 0 \Rightarrow g(y, z) = h(z)$. Thus $f(x, y, z) = ye^{xz} + h(z)$ and $f_z(x, y, z) = xye^{xz} + h'(z)$. But $f_z(x, y, z) = xye^{xz}$, so $h'(z) = 0 \Rightarrow h(z) = K$. Hence $f(x, y, z) = ye^{xz}$ (taking $K = 0$).

(b) $\mathbf{r}(0) = \langle 1, -1, 0 \rangle$, $\mathbf{r}(2) = \langle 5, 3, 0 \rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(5, 3, 0) - f(1, -1, 0) = 3e^0 + e^0 = 4$.

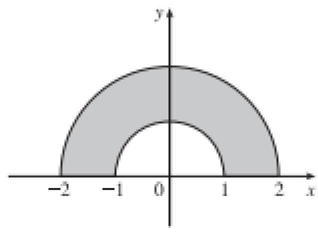
20. The functions $\sin y$ and $x \cos y - \sin y$ have continuous first-order derivatives on $\mathbb{R}^2$ and $\dfrac{\partial}{\partial y}(\sin y) = \cos y = \dfrac{\partial}{\partial x}(x \cos y - \sin y)$, so $\mathbf{F}(x, y) = \sin y\, \mathbf{i} + (x \cos y - \sin y)\, \mathbf{j}$ is a conservative vector field by Theorem 6 and hence the line integral is independent of path. Thus a potential function f exists, and $f_x(x, y) = \sin y$ implies $f(x, y) = x \sin y + g(y)$ and $f_y(x, y) = x \cos y + g'(y)$. But $f_y(x, y) = x \cos y - \sin y$ so $g'(y) = -\sin y \Rightarrow g(y) = \cos y + K$. We can take $K = 0$, so $f(x, y) = x \sin y + \cos y$. Then $\int_C \sin y\, dx + (x \cos y - \sin y)\, dy = f(1, \pi) - f(2, 0) = -1 - 1 = -2$.

21. If $\mathbf{F}$ is conservative, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path. This means that the work done along all piecewise-smooth curves that have the described initial and terminal points is the same. Your reply: It doesn't matter which curve is chosen.

24. $\mathbf{F}(x,y) = e^{-y}\,\mathbf{i} - xe^{-y}\,\mathbf{j}$, $W = \int_C \mathbf{F}\cdot d\mathbf{r}$. Since $\dfrac{\partial}{\partial y}(e^{-y}) = -e^{-y} = \dfrac{\partial}{\partial x}(-xe^{-y})$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x = e^{-y} \;\Rightarrow\; f(x,y) = xe^{-y} + g(y) \;\Rightarrow\; f_y = -xe^{-y} + g'(y) \;\Rightarrow\; g'(y) = 0$, so we can take $f(x,y) = xe^{-y}$ as a potential function for $\mathbf{F}$. Thus $W = \int_C \mathbf{F}\cdot d\mathbf{r} = f(2,0) - f(0,1) = 2 - 0 = 2$.

26. If a vector field $\mathbf{F}$ is conservative, then around any closed path C, $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$. For any closed path we draw in the field, it appears that some vectors on the curve point in approximately the same direction as the curve and a similar number point in roughly the opposite direction. (Some appear perpendicular to the curve as well.) Therefore it is plausible that $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed curve C which means $\mathbf{F}$ is conservative.

33. $D = \{(x,y) \mid 1 \le x^2 + y^2 \le 4,\ y \ge 0\}$ is the semiannular region in the upper half-plane between circles centered at the origin of radii 1 and 2 (including all boundary points).



(a) D includes boundary points, so it is not open. [Note that at any boundary point, $(1,0)$ for instance, any disk centered there cannot lie entirely in D.]

(b) The region consists of one piece, so it's connected.

(c) D is connected and has no holes, so it's simply-connected.