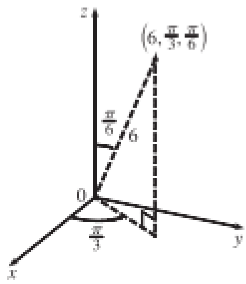
Section 15.9 - 1, 4, 5, 10, 13, 17, 19, 27, 35, 39

1. (a)

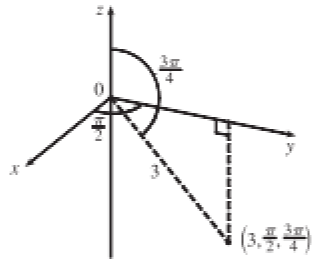


From Equations 1, $x = \rho \sin\phi \cos\theta = 6 \sin\frac{\pi}{6} \cos\frac{\pi}{3} = 6 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{2}$,

$y = \rho \sin\phi \sin\theta = 6 \sin\frac{\pi}{6} \sin\frac{\pi}{3} = 6 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$, and

$z = \rho \cos\phi = 6 \cos\frac{\pi}{6} = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$, so the point is $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 3\sqrt{3}\right)$ in rectangular coordinates.

(b)



$x = 3 \sin\frac{3\pi}{4} \cos\frac{\pi}{2} = 3 \cdot \frac{\sqrt{2}}{2} \cdot 0 = 0$,

$y = 3 \sin\frac{3\pi}{4} \sin\frac{\pi}{2} = 3 \cdot \frac{\sqrt{2}}{2} \cdot 1 = \frac{3\sqrt{2}}{2}$, and

$z = 3 \cos\frac{3\pi}{4} = 3\left(-\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{2}$, so the point is $\left(0, \frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$ in rectangular coordinates.

4. (a) $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 0 + 3} = 2$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{\sqrt{3}}{2} \Rightarrow \phi = \dfrac{\pi}{6}$, and $\cos\theta = \dfrac{x}{\rho \sin\phi} = \dfrac{1}{2\sin(\pi/6)} = 1 \Rightarrow$ $\theta = 0$. Thus spherical coordinates are $\left(2, 0, \dfrac{\pi}{6}\right)$.

(b) $\rho = \sqrt{3 + 1 + 12} = 4$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2} \Rightarrow \phi = \dfrac{\pi}{6}$, and $\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{\sqrt{3}}{4\sin(\pi/6)} = \dfrac{\sqrt{3}}{2} \Rightarrow$ $\theta = \dfrac{11\pi}{6}$ [since $y < 0$]. Thus spherical coordinates are $\left(4, \dfrac{11\pi}{6}, \dfrac{\pi}{6}\right)$.

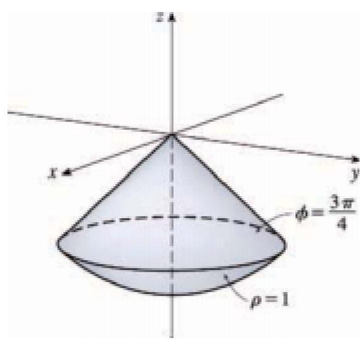
5. Since $\phi = \frac{\pi}{3}$, the surface is the top half of the right circular cone with vertex at the origin and axis the positive z-axis.

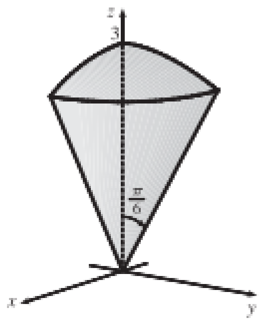
10. (a) $x^2 - 2x + y^2 + z^2 = 0 \Leftrightarrow (x^2 + y^2 + z^2) - 2x = 0 \Leftrightarrow \rho^2 - 2(\rho \sin\phi \cos\theta) = 0$ or $\rho = 2\sin\phi\cos\theta$.

(b) $x + 2y + 3z = 1 \Leftrightarrow \rho\sin\phi\cos\theta + 2\rho\sin\phi\sin\theta + 3\rho\cos\phi = 1$ or $\rho = 1/(\sin\phi\cos\theta + 2\sin\phi\sin\theta + 3\cos\phi)$.

13. $\rho \leq 1$ represents the solid sphere of radius 1 centered at the origin.

$\frac{3\pi}{4} \leq \phi \leq \pi$ restricts the solid to that portion on or below the cone $\phi = \frac{3\pi}{4}$.



17. 

The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 3, 0 \le \theta \le \pi/2, 0 \le \phi \le \pi/6\}$. This represents the solid region in the first octant bounded above by the sphere $\rho = 3$ and below by the cone $\phi = \pi/6$.

$$\begin{aligned}\int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi &= \int_0^{\pi/6} \sin\phi \, d\phi \int_0^{\pi/2} d\theta \int_0^3 \rho^2 \, d\rho \\ &= \left[-\cos\phi\right]_0^{\pi/6} \left[\theta\right]_0^{\pi/2} \left[\tfrac{1}{3}\rho^3\right]_0^3 \\ &= \left(1 - \frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{2}\right)(9) = \frac{9\pi}{4}(2 - \sqrt{3})\end{aligned}$$

19. The solid E is most conveniently described if we use cylindrical coordinates:

$E = \left\{(r, \theta, z) \mid 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 3, 0 \le z \le 2\right\}$. Then

$\iiint_E f(x, y, z)\, dV = \int_0^{\pi/2}\int_0^3\int_0^2 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta.$

27. The solid region is given by $E = \left\{(\rho, \theta, \phi) \mid 0 \le \rho \le a, 0 \le \theta \le 2\pi, \frac{\pi}{6} \le \phi \le \frac{\pi}{3}\right\}$ and its volume is

$$\begin{aligned}V = \iiint_E dV &= \int_{\pi/6}^{\pi/3}\int_0^{2\pi}\int_0^a \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_{\pi/6}^{\pi/3} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^a \rho^2 \, d\rho \\ &= \left[-\cos\phi\right]_{\pi/6}^{\pi/3} \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{3}\rho^3\right]_0^a = \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\right)(2\pi)\left(\tfrac{1}{3}a^3\right) = \tfrac{\sqrt{3}-1}{3}\pi a^3\end{aligned}$$

35. In spherical coordinates $z = \sqrt{x^2 + y^2}$ becomes $\cos\phi = \sin\phi$ or $\phi = \frac{\pi}{4}$. Then

$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \, d\phi \int_0^1 \rho^2 \, d\rho = 2\pi\left(-\frac{\sqrt{2}}{2} + 1\right)\left(\frac{1}{3}\right) = \frac{1}{3}\pi(2 - \sqrt{2}),$

$M_{xy} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^3 \sin\phi\cos\phi \, d\rho \, d\phi \, d\theta = 2\pi\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/4}\left(\frac{1}{4}\right) = \frac{\pi}{8}$ and by symmetry $M_{yz} = M_{xz} = 0$.

Hence $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \dfrac{3}{8(2 - \sqrt{2})}\right)$.

39. The region E of integration is the region above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 2$ in the first octant. Because E is in the first octant we have $0 \le \theta \le \frac{\pi}{2}$. The cone has equation $\phi = \frac{\pi}{4}$ (as in Example 4), so $0 \le \phi \le \frac{\pi}{4}$, and $0 \le \rho \le \sqrt{2}$. So the integral becomes

$$\begin{aligned}&\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)\,\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\ &\quad = \int_0^{\pi/4} \sin^3\phi \, d\phi \int_0^{\pi/2} \sin\theta\cos\theta \, d\theta \int_0^{\sqrt{2}} \rho^4 \, d\rho = \left(\int_0^{\pi/4}(1 - \cos^2\phi)\sin\phi \, d\phi\right)\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2}\left[\tfrac{1}{5}\rho^5\right]_0^{\sqrt{2}} \\ &\quad = \left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi/4} \cdot \tfrac{1}{2} \cdot \tfrac{1}{5}\left(\sqrt{2}\right)^5 = \left[\tfrac{\sqrt{2}}{12} - \tfrac{\sqrt{2}}{2} - \left(\tfrac{1}{3} - 1\right)\right] \cdot \tfrac{2\sqrt{2}}{5} = \tfrac{4\sqrt{2} - 5}{15}\end{aligned}$$