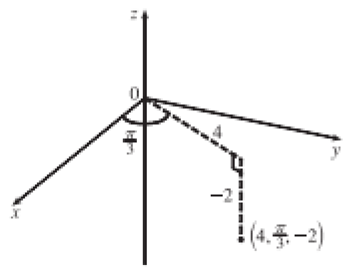
Section 15.8 - 1, 4, 5, 8, 9, 11, 20, 23, 29

1. (a)

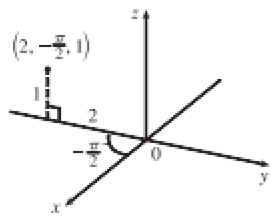


From Equations 1, $x = r\cos\theta = 4\cos\frac{\pi}{3} = 4 \cdot \frac{1}{2} = 2$,

$y = r\sin\theta = 4\sin\frac{\pi}{3} = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$, $z = -2$, so the point is

$\left(2, 2\sqrt{3}, -2\right)$ in rectangular coordinates.

(b)



$x = 2\cos\left(-\frac{\pi}{2}\right) = 0$, $y = 2\sin\left(-\frac{\pi}{2}\right) = -2$,

and $z = 1$, so the point is $(0, -2, 1)$ in rectangular coordinates.

4. (a) $r^2 = \left(2\sqrt{3}\right)^2 + 2^2 = 16$ so $r = 4$; $\tan\theta = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$ and the point $\left(2\sqrt{3}, 2\right)$ is in the first quadrant of the xy-plane, so $\theta = \frac{\pi}{6} + 2n\pi$; $z = -1$. Thus, one set of cylindrical coordinates is $\left(4, \frac{\pi}{6}, -1\right)$.

(b) $r^2 = 4^2 + (-3)^2 = 25$ so $r = 5$; $\tan\theta = \frac{-3}{4}$ and the point $(4, -3)$ is in the fourth quadrant of the xy-plane, so $\theta = \tan^{-1}\left(-\frac{3}{4}\right) + 2n\pi \approx -0.64 + 2n\pi$; $z = 2$. Thus, one set of cylindrical coordinates is $\left(5, \tan^{-1}\left(-\frac{3}{4}\right) + 2\pi, 2\right) \approx (5, 5.64, 2)$.

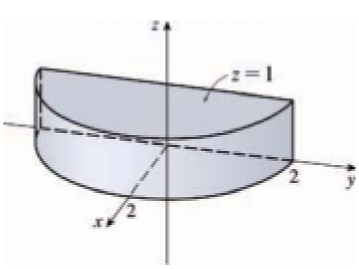
5. Since $\theta = \frac{\pi}{4}$ but r and z may vary, the surface is a vertical half-plane including the z-axis and intersecting the xy-plane in the half-line $y = x$, $x \geq 0$.

8. Since $2r^2 + z^2 = 1$ and $r^2 = x^2 + y^2$, we have $2(x^2 + y^2) + z^2 = 1$ or $2x^2 + 2y^2 + z^2 = 1$, an ellipsoid centered at the origin with intercepts $x = \pm\frac{1}{\sqrt{2}}$, $y = \pm\frac{1}{\sqrt{2}}$, $z = \pm 1$.

9. (a) Substituting $x^2 + y^2 = r^2$ and $x = r\cos\theta$, the equation $x^2 - x + y^2 + z^2 = 1$ becomes $r^2 - r\cos\theta + z^2 = 1$ or $z^2 = 1 + r\cos\theta - r^2$.

(b) Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the equation $z = x^2 - y^2$ becomes $z = (r\cos\theta)^2 - (r\sin\theta)^2 = r^2(\cos^2\theta - \sin^2\theta)$ or $z = r^2\cos 2\theta$.

11.



$0 \leq r \leq 2$ and $0 \leq z \leq 1$ describe a solid circular cylinder with radius 2, axis the z-axis, and height 1, but $-\pi/2 \leq \theta \leq \pi/2$ restricts the solid to the first and fourth quadrants of the xy-plane, so we have a half-cylinder.

20. In cylindrical coordinates E is bounded by the planes $z = 0$, $z = r\cos\theta + r\sin\theta + 5$ and the cylinders $r = 2$ and $r = 3$, so E is given by $\{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 2 \leq r \leq 3, 0 \leq z \leq r\cos\theta + r\sin\theta + 5\}$. Thus

$$\iiint_E x\,dV = \int_0^{2\pi}\int_2^3\int_0^{r\cos\theta + r\sin\theta+5}(r\cos\theta)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_2^3(r^2\cos\theta)[\,z\,]_{z=0}^{z=r\cos\theta+r\sin\theta+5}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^3(r^2\cos\theta)(r\cos\theta + r\sin\theta + 5)\,dr\,d\theta = \int_0^{2\pi}\int_2^3(r^3(\cos^2\theta + \cos\theta\sin\theta) + 5r^2\cos\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\tfrac{1}{4}r^4(\cos^2\theta + \cos\theta\sin\theta) + \tfrac{5}{3}r^3\cos\theta\right]_{r=2}^{r=3}d\theta$$

$$= \int_0^{2\pi}\left[\left(\tfrac{81}{4} - \tfrac{16}{4}\right)(\cos^2\theta + \cos\theta\sin\theta) + \tfrac{5}{3}(27-8)\cos\theta\right]d\theta$$

$$= \int_0^{2\pi}\left(\tfrac{65}{4}\left(\tfrac{1}{2}(1+\cos 2\theta) + \cos\theta\sin\theta\right) + \tfrac{95}{3}\cos\theta\right)d\theta = \left[\tfrac{65}{8}\theta + \tfrac{65}{16}\sin 2\theta + \tfrac{65}{8}\sin^2\theta + \tfrac{95}{3}\sin\theta\right]_0^{2\pi} = \tfrac{65}{4}\pi$$

23. In cylindrical coordinates, E is bounded below by the cone $z = r$ and above by the sphere $r^2 + z^2 = 2$ or $z = \sqrt{2 - r^2}$. The cone and the sphere intersect when $2r^2 = 2 \quad \Rightarrow \quad r = 1$, so $E = \left\{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, r \leq z \leq \sqrt{2-r^2}\right\}$ and the volume is

$$\iiint_E dV = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 [rz]_{z=r}^{z=\sqrt{2-r^2}}\,dr\,d\theta = \int_0^{2\pi}\int_0^1\left(r\sqrt{2-r^2} - r^2\right)dr\,d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^1\left(r\sqrt{2-r^2} - r^2\right)dr = 2\pi\left[-\tfrac{1}{3}(2-r^2)^{3/2} - \tfrac{1}{3}r^3\right]_0^1$$

$$= 2\pi\left(-\tfrac{1}{3}\right)\left(1 + 1 - 2^{3/2}\right) = -\tfrac{2}{3}\pi\left(2 - 2\sqrt{2}\right) = \tfrac{4}{3}\pi\left(\sqrt{2} - 1\right)$$

29. The region of integration is the region above the cone $z = \sqrt{x^2 + y^2}$, or $z = r$, and below the plane $z = 2$. Also, we have $-2 \leq y \leq 2$ with $-\sqrt{4-y^2} \leq x \leq \sqrt{4-y^2}$ which describes a circle of radius 2 in the xy-plane centered at $(0, 0)$. Thus,

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy = \int_0^{2\pi}\int_0^2\int_r^2(r\cos\theta)\,z\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\int_r^2 r^2(\cos\theta)\,z\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^2(\cos\theta)\left[\tfrac{1}{2}z^2\right]_{z=r}^{z=2}dr\,d\theta = \tfrac{1}{2}\int_0^{2\pi}\int_0^2 r^2(\cos\theta)\left(4 - r^2\right)dr\,d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi}\cos\theta\,d\theta\int_0^2\left(4r^2 - r^4\right)dr = \tfrac{1}{2}\left[\sin\theta\right]_0^{2\pi}\left[\tfrac{4}{3}r^3 - \tfrac{1}{5}r^5\right]_0^2 = 0$$