Section 15.7 - 7, 13, 17, 20, 27, 30, 33, 36

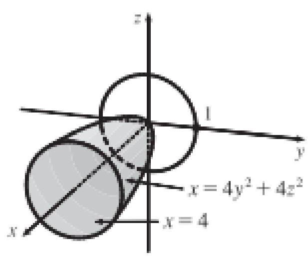
7. $\int_0^{\pi/2}\int_0^y\int_0^x \cos(x+y+z)\,dz\,dx\,dy = \int_0^{\pi/2}\int_0^y \left[\sin(x+y+z)\right]_{z=0}^{z=x} dx\,dy$

$$= \int_0^{\pi/2}\int_0^y [\sin(2x+y)-\sin(x+y)]\,dx\,dy$$
$$= \int_0^{\pi/2}\left[-\tfrac{1}{2}\cos(2x+y)+\cos(x+y)\right]_{x=0}^{x=y} dy$$
$$= \int_0^{\pi/2}\left[-\tfrac{1}{2}\cos 3y+\cos 2y+\tfrac{1}{2}\cos y-\cos y\right] dy$$
$$= \left[-\tfrac{1}{6}\sin 3y+\tfrac{1}{2}\sin 2y-\tfrac{1}{2}\sin y\right]_0^{\pi/2} = \tfrac{1}{6}-\tfrac{1}{2} = -\tfrac{1}{3}$$

13. Here $E=\{(x,y,z)\mid 0\le x\le 1, 0\le y\le \sqrt{x}, 0\le z\le 1+x+y\}$, so

$$\iiint_E 6xy\,dV = \int_0^1\int_0^{\sqrt{x}}\int_0^{1+x+y} 6xy\,dz\,dy\,dx = \int_0^1\int_0^{\sqrt{x}}\left[6xyz\right]_{z=0}^{z=1+x+y} dy\,dx$$
$$= \int_0^1\int_0^{\sqrt{x}} 6xy(1+x+y)\,dy\,dx = \int_0^1\left[3xy^2+3x^2y^2+2xy^3\right]_{y=0}^{y=\sqrt{x}} dx$$
$$= \int_0^1 (3x^2+3x^3+2x^{5/2})\,dx = \left[x^3+\tfrac{3}{4}x^4+\tfrac{4}{7}x^{7/2}\right]_0^1 = \tfrac{65}{28}$$

17.

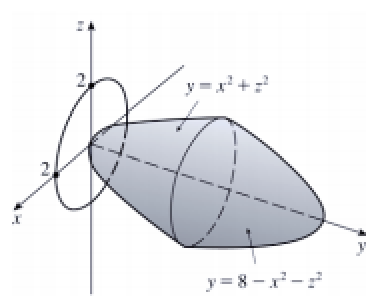



The projection of $E$ on the $yz$-plane is the disk $y^2+z^2\le 1$. Using polar coordinates $y=r\cos\theta$ and $z=r\sin\theta$, we get

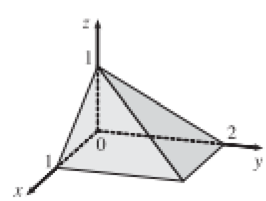
$$\iiint_E x\,dV = \iint_D\left[\int_{4y^2+4z^2}^4 x\,dx\right] dA = \tfrac{1}{2}\iint_D\left[4^2-(4y^2+4z^2)^2\right] dA$$
$$= 8\int_0^{2\pi}\int_0^1 (1-r^4)\,r\,dr\,d\theta = 8\int_0^{2\pi} d\theta \int_0^1 (r-r^5)\,dr$$
$$= 8(2\pi)\left[\tfrac{1}{2}r^2-\tfrac{1}{6}r^6\right]_0^1 = \tfrac{16\pi}{3}$$

20. The paraboloids intersect when $x^2+z^2=8-x^2-z^2 \;\Leftrightarrow\; x^2+z^2=4$, thus the intersection is the circle $x^2+z^2=4$, $y=4$. The projection of $E$ onto the $xz$-plane is the disk $x^2+z^2\le 4$, so $E=\left\{(x,y,z)\mid x^2+z^2\le y\le 8-x^2-z^2, x^2+z^2\le 4\right\}$. Let $D=\left\{(x,z)\mid x^2+z^2\le 4\right\}$. Then using polar coordinates $x=r\cos\theta$ and $z=r\sin\theta$, we have

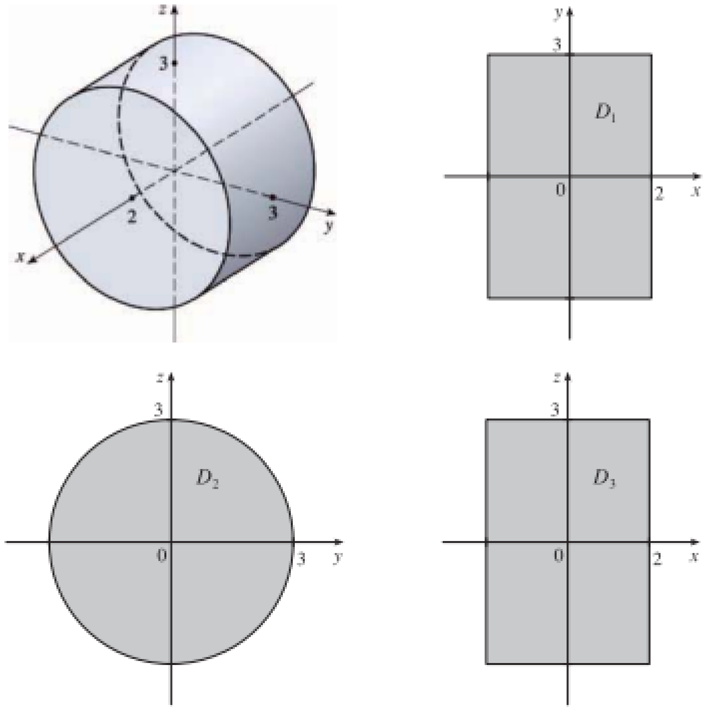
$$V=\iiint_E dV = \iint_D\left(\int_{x^2+z^2}^{8-x^2-z^2} dy\right) dA = \iint_D (8-2x^2-2z^2)\,dA$$
$$= \int_0^{2\pi}\int_0^2 (8-2r^2)\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 (8r-2r^3)\,dr$$
$$= \left[\theta\right]_0^{2\pi}\left[4r^2-\tfrac{1}{2}r^4\right]_0^2 = 2\pi(16-8) = 16\pi$$




27. $E=\{(x,y,z)\mid 0\le x\le 1, 0\le z\le 1-x, 0\le y\le 2-2z\}$, the solid bounded by the three coordinate planes and the planes $z=1-x$, $y=2-2z$.



**30.**




If $D_1$, $D_2$, $D_3$ are the projections of $E$ on the $xy$-, $yz$-, and $xz$-planes, then

$$D_1 = \{(x, y) \mid -2 \le x \le 2,\ -3 \le y \le 3\}$$
$$D_2 = \{(y, z) \mid y^2 + z^2 \le 9\}$$
$$D_3 = \{(x, z) \mid -2 \le x \le 2,\ -3 \le z \le 3\}$$

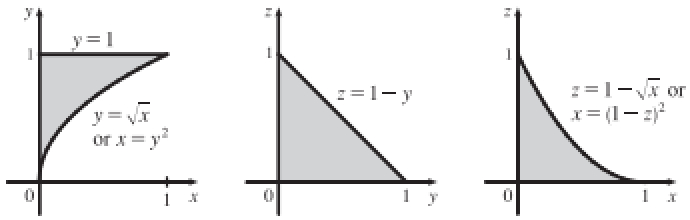
Therefore

$$\begin{aligned}
E &= \left\{(x, y, z) \mid -2 \le x \le 2,\ -3 \le y \le 3,\ -\sqrt{9-y^2} \le z \le \sqrt{9-y^2}\right\} \\
&= \left\{(x, y, z) \mid -3 \le y \le 3,\ -\sqrt{9-y^2} \le z \le \sqrt{9-y^2},\ -2 \le x \le 2\right\} \\
&= \left\{(x, y, z) \mid -3 \le z \le 3,\ -\sqrt{9-z^2} \le y \le \sqrt{9-z^2},\ -2 \le x \le 2\right\} \\
&= \left\{(x, y, z) \mid -2 \le x \le 2,\ -3 \le z \le 3,\ -\sqrt{9-z^2} \le y \le \sqrt{9-z^2}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\iiint_E f(x, y, z)\, dV &= \int_{-2}^{2} \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y, z)\, dz\, dy\, dx = \int_{-3}^{3} \int_{-2}^{2} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y, z)\, dz\, dx\, dy \\
&= \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{-2}^{2} f(x, y, z)\, dx\, dz\, dy = \int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \int_{-2}^{2} f(x, y, z)\, dx\, dy\, dz \\
&= \int_{-2}^{2} \int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x, y, z)\, dy\, dz\, dx = \int_{-3}^{3} \int_{-2}^{2} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x, y, z)\, dy\, dx\, dz
\end{aligned}$$

**33.**



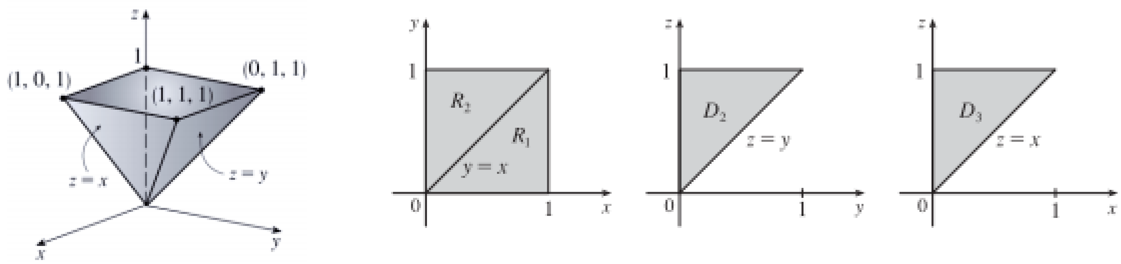

The diagrams show the projections of $E$ on the $xy$-, $yz$-, and $xz$-planes. Therefore

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy = \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$$

**36.**




$\int_0^1\int_y^1\int_0^z f(x,y,z)\,dx\,dz\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le x \le z, y \le z \le 1, 0 \le y \le 1\}$.

Notice that $E$ is bounded below by two different surfaces, so we must split the projection of $E$ onto the $xy$-plane into two regions as in the second diagram. If $D_1$, $D_2$, and $D_3$ are the projections of $E$ on the $xy$-, $yz$- and $xz$-planes then

$$D_1 = R_1 \cup R_2 = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\} \cup \{(x,y) \mid 0 \le x \le 1, x \le y \le 1\}$$
$$= \{(x,y) \mid 0 \le y \le 1, y \le x \le 1\} \cup \{(x,y) \mid 0 \le y \le 1, 0 \le x \le y\},$$

$D_2 = \{(y,z) \mid 0 \le y \le 1, y \le z \le 1\} = \{(y,z) \mid 0 \le z \le 1, 0 \le y \le z\}$, and

$D_3 = \{(x,z) \mid 0 \le x \le 1, x \le z \le 1\} = \{(x,z) \mid 0 \le z \le 1, 0 \le x \le z\}$.

Thus we also have

$$E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, x \le z \le 1\} \cup \{(x,y,z) \mid 0 \le x \le 1, x \le y \le 1, y \le z \le 1\}$$
$$= \{(x,y,z) \mid 0 \le y \le 1, y \le x \le 1, x \le z \le 1\} \cup \{(x,y,z) \mid 0 \le y \le 1, 0 \le x \le y, y \le z \le 1\}$$
$$= \{(x,y,z) \mid 0 \le z \le 1, 0 \le y \le z, 0 \le x \le z\} = \{(x,y,z) \mid 0 \le x \le 1, x \le z \le 1, 0 \le y \le z\}$$
$$= \{(x,y,z) \mid 0 \le z \le 1, 0 \le x \le z, 0 \le y \le z\}.$$

Then

$$\int_0^1\int_y^1\int_0^z f(x,y,z)\,dx\,dz\,dy = \int_0^1\int_0^x\int_x^1 f(x,y,z)\,dz\,dy\,dx + \int_0^1\int_x^1\int_y^1 f(x,y,z)\,dz\,dy\,dx$$
$$= \int_0^1\int_y^1\int_x^1 f(x,y,z)\,dz\,dx\,dy + \int_0^1\int_0^y\int_y^1 f(x,y,z)\,dz\,dx\,dy$$
$$= \int_0^1\int_0^z\int_0^z f(x,y,z)\,dx\,dy\,dz = \int_0^1\int_x^1\int_0^z f(x,y,z)\,dy\,dz\,dx$$
$$= \int_0^1\int_0^z\int_0^z f(x,y,z)\,dy\,dx\,dz$$