Section 15.6 - 4, 5, 7, 11, 13

4. $z = f(x,y) = 1 + 3x + 2y^2$ with $0 \le x \le 2y$, $0 \le y \le 1$. Thus by Formula 2,

$$A(S) = \iint_D \sqrt{1+(3)^2+(4y)^2}\,dA = \int_0^1\int_0^{2y}\sqrt{10+16y^2}\,dx\,dy = \int_0^1\sqrt{10+16y^2}\,\big[\,x\,\big]_{x=0}^{x=2y}\,dy$$
$$= \int_0^1 2y\sqrt{10+16y^2}\,dy = 2\cdot\tfrac{1}{32}\cdot\tfrac{2}{3}(10+16y^2)^{3/2}\Big]_0^1 = \tfrac{1}{24}(26^{3/2}-10^{3/2})$$

5. $y^2 + z^2 = 9 \;\Rightarrow\; z = \sqrt{9-y^2}$. $f_x = 0$, $f_y = -y(9-y^2)^{-1/2} \;\Rightarrow$

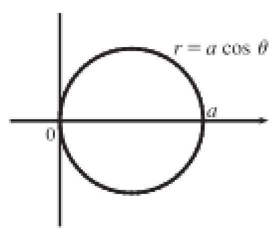
$$A(S) = \int_0^4\int_0^2 \sqrt{0^2 + [-y(9-y^2)^{-1/2}]^2 + 1}\,dy\,dx = \int_0^4\int_0^2\sqrt{\frac{y^2}{9-y^2}+1}\,dy\,dx$$
$$= \int_0^4\int_0^2 \frac{3}{\sqrt{9-y^2}}\,dy\,dx = 3\int_0^4\left[\sin^{-1}\frac{y}{3}\right]_{y=0}^{y=2}dx = 3\left[\left(\sin^{-1}\left(\tfrac{2}{3}\right)\right)x\right]_0^4 = 12\sin^{-1}\left(\tfrac{2}{3}\right)$$

7. $z = f(x,y) = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$A(S) = \iint_D \sqrt{1+4x^2+4y^2}\,dA = \int_0^{2\pi}\int_1^2\sqrt{1+4r^2}\,r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_1^2 r\sqrt{1+4r^2}\,dr$$
$$= \big[\,\theta\,\big]_0^{2\pi}\left[\tfrac{1}{12}(1+4r^2)^{3/2}\right]_1^2 = \tfrac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)$$

11. $z = \sqrt{a^2-x^2-y^2}$, $z_x = -x(a^2-x^2-y^2)^{-1/2}$, $z_y = -y(a^2-x^2-y^2)^{-1/2}$,

$$A(S) = \iint_D \sqrt{\frac{x^2+y^2}{a^2-x^2-y^2}+1}\,dA$$
$$= \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\sqrt{\frac{r^2}{a^2-r^2}+1}\,r\,dr\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\frac{ar}{\sqrt{a^2-r^2}}\,dr\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\left[-a\sqrt{a^2-r^2}\right]_{r=0}^{r=a\cos\theta}d\theta$$
$$= \int_{-\pi/2}^{\pi/2} -a\left(\sqrt{a^2-a^2\cos^2\theta}-a\right)d\theta = 2a^2\int_0^{\pi/2}\left(1-\sqrt{1-\cos^2\theta}\right)d\theta$$
$$= 2a^2\int_0^{\pi/2}d\theta - 2a^2\int_0^{\pi/2}\sqrt{\sin^2\theta}\,d\theta = a^2\pi - 2a^2\int_0^{\pi/2}\sin\theta\,d\theta = a^2(\pi-2)$$



13. $z = f(x,y) = e^{-x^2-y^2}$, $f_x = -2xe^{-x^2-y^2}$, $f_y = -2ye^{-x^2-y^2}$. Then

$$A(S) = \iint_{x^2+y^2\le 4}\sqrt{(-2xe^{-x^2-y^2})^2+(-2ye^{-x^2-y^2})^2+1}\,dA = \iint_{x^2+y^2\le 4}\sqrt{4(x^2+y^2)e^{-2(x^2+y^2)}+1}\,dA.$$

Converting to polar coordinates we have

$$A(S) = \int_0^{2\pi}\int_0^2\sqrt{4r^2e^{-2r^2}+1}\,r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^2 r\sqrt{4r^2e^{-2r^2}+1}\,dr$$
$$= 2\pi\int_0^2 r\sqrt{4r^2e^{-2r^2}+1}\,dr \approx 13.9783 \text{ using a calculator.}$$