Section 15.5 - 2, 5, 9, 13, 15

2. $Q = \iint_D \sigma(x,y)\,dA = \iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{r^2}\,r\,dr\,d\theta$
$= \int_0^{2\pi} d\theta\ \int_0^1 r^2\,dr = [\theta]_0^{2\pi}\ \left[\frac{1}{3}r^3\right]_0^1 = 2\pi\cdot\frac{1}{3} = \frac{2\pi}{3}$ C

5. $m = \int_0^2\int_{x/2}^{3-x}(x+y)\,dy\,dx = \int_0^2\left[xy+\frac{1}{2}y^2\right]_{y=x/2}^{y=3-x}dx = \int_0^2\left[x\left(3-\frac{3}{2}x\right)+\frac{1}{2}(3-x)^2-\frac{1}{8}x^2\right]dx$
$= \int_0^2\left(-\frac{9}{8}x^2+\frac{9}{2}\right)dx = \left[-\frac{9}{8}\left(\frac{1}{3}x^3\right)+\frac{9}{2}x\right]_0^2 = 6,$

$M_y = \int_0^2\int_{x/2}^{3-x}(x^2+xy)\,dy\,dx = \int_0^2\left[x^2y+\frac{1}{2}xy^2\right]_{y=x/2}^{y=3-x}\,dx = \int_0^2\left(\frac{9}{2}x-\frac{9}{8}x^3\right)dx = \frac{9}{2},$

$M_x = \int_0^2\int_{x/2}^{3-y}(xy+y^2)\,dy\,dx = \int_0^2\left[\frac{1}{2}xy^2+\frac{1}{3}y^3\right]_{y=x/2}^{y=3-x}\,dx = \int_0^2\left(9-\frac{9}{2}x\right)dx = 9.$

Hence $m=6$, $(\overline{x},\overline{y}) = \left(\frac{M_y}{m},\frac{M_x}{m}\right) = \left(\frac{3}{4},\frac{3}{2}\right)$.

9. Note that $\sin(\pi x/L)\geq 0$ for $0\leq x\leq L$.

$m = \int_0^L\int_0^{\sin(\pi x/L)} y\,dy\,dx = \int_0^L \frac{1}{2}\sin^2(\pi x/L)\,dx = \frac{1}{2}\left[\frac{1}{2}x-\frac{L}{4\pi}\sin(2\pi x/L)\right]_0^L = \frac{1}{4}L,$

$M_y = \int_0^L\int_0^{\sin(\pi x/L)} x\cdot y\,dy\,dx = \frac{1}{2}\int_0^L x\sin^2(\pi x/L)\,dx \quad \begin{bmatrix}\text{integrate by parts with}\\ u=x,\ dv=\sin^2(\pi x/L)\,dx\end{bmatrix}$

$= \frac{1}{2}\cdot x\left(\frac{1}{2}x-\frac{L}{4\pi}\sin(2\pi x/L)\right)\Big]_0^L - \frac{1}{2}\int_0^L\left[\frac{1}{2}x-\frac{L}{4\pi}\sin(2\pi x/L)\right]dx$

$= \frac{1}{4}L^2-\frac{1}{2}\left[\frac{1}{4}x^2+\frac{L^2}{4\pi^2}\cos(2\pi x/L)\right]_0^L = \frac{1}{4}L^2-\frac{1}{2}\left(\frac{1}{4}L^2+\frac{L^2}{4\pi^2}-\frac{L^2}{4\pi^2}\right) = \frac{1}{8}L^2,$

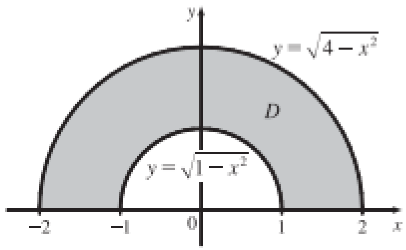
$M_x = \int_0^L\int_0^{\sin(\pi x/L)} y\cdot y\,dy\,dx = \int_0^L\frac{1}{3}\sin^3(\pi x/L)\,dx = \frac{1}{3}\int_0^L\left[1-\cos^2(\pi x/L)\right]\sin(\pi x/L)\,dx$

[substitute $u=\cos(\pi x/L)$] $\Rightarrow$ $du = -\frac{\pi}{L}\sin(\pi x/L)$]

$= \frac{1}{3}\left(-\frac{L}{\pi}\right)\left[\cos(\pi x/L)-\frac{1}{3}\cos^3(\pi x/L)\right]_0^L = -\frac{L}{3\pi}\left(-1+\frac{1}{3}-1+\frac{1}{3}\right) = \frac{4}{9\pi}L.$

Hence $m=\frac{L}{4}$, $(\overline{x},\overline{y}) = \left(\frac{L^2/8}{L/4},\frac{4L/(9\pi)}{L/4}\right) = \left(\frac{L}{2},\frac{16}{9\pi}\right)$.

13.



$\rho(x,y) = k\sqrt{x^2+y^2} = kr,$

$m = \iint_D \rho(x,y)dA = \int_0^{\pi}\int_1^2 kr \cdot r\,dr\,d\theta$

$= k\int_0^{\pi} d\theta \int_1^2 r^2\,dr = k(\pi)\left[\frac{1}{3}r^3\right]_1^2 = \frac{7}{3}\pi k,$

$$M_y = \iint_D x\rho(x,y)dA = \int_0^{\pi}\int_1^2 (r\cos\theta)(kr)\,r\,dr\,d\theta = k\int_0^{\pi}\cos\theta\,d\theta \int_1^2 r^3\,dr$$

$$= k\left[\sin\theta\right]_0^{\pi}\left[\frac{1}{4}r^4\right]_1^2 = k(0)\left(\frac{15}{4}\right) = 0$$ [this is to be expected as the region and density function are symmetric about the y-axis]

$$M_x = \iint_D y\rho(x,y)dA = \int_0^{\pi}\int_1^2 (r\sin\theta)(kr)\,r\,dr\,d\theta = k\int_0^{\pi}\sin\theta\,d\theta \int_1^2 r^3\,dr$$

$$= k\left[-\cos\theta\right]_0^{\pi}\left[\frac{1}{4}r^4\right]_1^2 = k(1+1)\left(\frac{15}{4}\right) = \frac{15}{2}k.$$

Hence $(\overline{x},\overline{y}) = \left(0, \frac{15k/2}{7\pi k/3}\right) = \left(0, \frac{45}{14\pi}\right)$.

15. Placing the vertex opposite the hypotenuse at $(0,0)$, $\rho(x,y) = k(x^2+y^2)$. Then

$m = \int_0^a\int_0^{a-x} k(x^2+y^2)\,dy\,dx = k\int_0^a \left[ax^2 - x^3 + \frac{1}{3}(a-x)^3\right]dx = k\left[\frac{1}{3}ax^3 - \frac{1}{4}x^4 - \frac{1}{12}(a-x)^4\right]_0^a = \frac{1}{6}ka^4.$

By symmetry,

$$M_y = M_x = \int_0^a\int_0^{a-x} ky(x^2+y^2)\,dy\,dx = k\int_0^a\left[\frac{1}{2}(a-x)^2x^2 + \frac{1}{4}(a-x)^4\right]dx$$

$$= k\left[\frac{1}{6}a^2x^3 - \frac{1}{4}ax^4 + \frac{1}{10}x^5 - \frac{1}{20}(a-x)^5\right]_0^a = \frac{1}{15}ka^5$$

Hence $(\overline{x},\overline{y}) = \left(\frac{2}{5}a, \frac{2}{5}a\right)$.