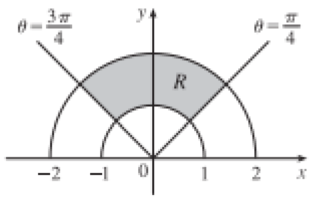
Section 15.4 - 4, 5, 9, 13, 21, 24, 31, 33

4. The region R is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 3 \le r \le 6,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$.

Thus $\iint_R f(x,y)\,dA = \int_{-\pi/2}^{\pi/2}\int_3^6 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

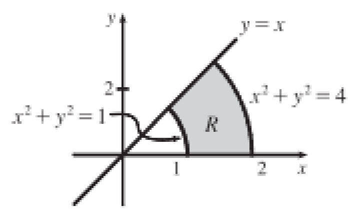
5. The integral $\int_{\pi/4}^{3\pi/4}\int_1^2 r\,dr\,d\theta$ represents the area of the region $R = \{(r,\theta) \mid 1 \le r \le 2,\ \pi/4 \le \theta \le 3\pi/4\}$, the top quarter portion of a ring (annulus).



$$\int_{\pi/4}^{3\pi/4}\int_1^2 r\,dr\,d\theta = \left(\int_{\pi/4}^{3\pi/4} d\theta\right)\left(\int_1^2 r\,dr\right)$$
$$= \left[\theta\right]_{\pi/4}^{3\pi/4}\left[\tfrac{1}{2}r^2\right]_1^2 = \left(\tfrac{3\pi}{4} - \tfrac{\pi}{4}\right)\cdot\tfrac{1}{2}(4-1) = \tfrac{\pi}{2}\cdot\tfrac{3}{2} = \tfrac{3\pi}{4}$$

9. $$\iint_R \sin(x^2+y^2)\,dA = \int_0^{\pi/2}\int_1^3 \sin(r^2)\,r\,dr\,d\theta = \left(\int_0^{\pi/2} d\theta\right)\left(\int_1^3 r\sin(r^2)\,dr\right)$$
$$= \left[\theta\right]_0^{\pi/2}\left[-\tfrac{1}{2}\cos(r^2)\right]_1^3$$
$$= \left(\tfrac{\pi}{2}\right)\left[-\tfrac{1}{2}(\cos 9 - \cos 1)\right] = \tfrac{\pi}{4}(\cos 1 - \cos 9)$$

13. R is the region shown in the figure, and can be described by $R = \{(r,\theta) \mid 0 \le \theta \le \pi/4, 1 \le r \le 2\}$. Thus $\iint_R \arctan(y/x)\,dA = \int_0^{\pi/4}\int_1^2 \arctan(\tan\theta)\,r\,dr\,d\theta$ since $y/x = \tan\theta$. Also, $\arctan(\tan\theta) = \theta$ for $0 \le \theta \le \pi/4$, so the integral becomes $\int_0^{\pi/4}\int_1^2 \theta\,r\,dr\,d\theta = \int_0^{\pi/4}\theta\,d\theta\int_1^2 r\,dr = \left[\tfrac{1}{2}\theta^2\right]_0^{\pi/4}\left[\tfrac{1}{2}r^2\right]_1^2 = \tfrac{\pi^2}{32}\cdot\tfrac{3}{2} = \tfrac{3}{64}\pi^2$.



21. The hyperboloid of two sheets $-x^2 - y^2 + z^2 = 1$ intersects the plane $z = 2$ when $-x^2 - y^2 + 4 = 1$ or $x^2 + y^2 = 3$. So the solid region lies above the surface $z = \sqrt{1 + x^2 + y^2}$ and below the plane $z = 2$ for $x^2 + y^2 \le 3$, and its volume is

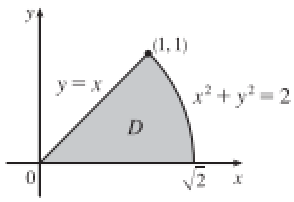
$$V = \iint_{x^2+y^2\le 3}\left(2 - \sqrt{1+x^2+y^2}\right)dA = \int_0^{2\pi}\int_0^{\sqrt{3}}\left(2 - \sqrt{1+r^2}\right)r\,dr\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{3}}\left(2r - r\sqrt{1+r^2}\right)dr = \left[\theta\right]_0^{2\pi}\left[r^2 - \tfrac{1}{3}(1+r^2)^{3/2}\right]_0^{\sqrt{3}}$$
$$= 2\pi\left(3 - \tfrac{8}{3} - 0 + \tfrac{1}{3}\right) = \tfrac{4}{3}\pi$$

24. The paraboloid $z = 1 + 2x^2 + 2y^2$ intersects the plane $z = 7$ when $7 = 1 + 2x^2 + 2y^2$ or $x^2 + y^2 = 3$ and we are restricted to the first octant, so

$$V = \iint\limits_{\substack{x^2+y^2 \le 3,\\ x\ge 0,\, y\ge 0}} \left[7 - \left(1 + 2x^2 + 2y^2\right)\right] dA = \int_0^{\pi/2} \int_0^{\sqrt{3}} \left[7 - \left(1 + 2r^2\right)\right] r\,dr\,d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^{\sqrt{3}} \left(6r - 2r^3\right) dr = \left[\,\theta\,\right]_0^{\pi/2} \left[3r^2 - \tfrac{1}{2}r^4\right]_0^{\sqrt{3}} = \tfrac{\pi}{2} \cdot \tfrac{9}{2} = \tfrac{9}{4}\pi$$

31.



$$\int_0^{\pi/4} \int_0^{\sqrt{2}} (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta = \int_0^{\pi/4} (\cos\theta + \sin\theta)\, d\theta \int_0^{\sqrt{2}} r^2\,dr$$

$$= [\sin\theta - \cos\theta]_0^{\pi/4} \left[\tfrac{1}{3}r^3\right]_0^{\sqrt{2}}$$

$$= \left[\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} - 0 + 1\right] \cdot \tfrac{1}{3}\left(2\sqrt{2} - 0\right) = \tfrac{2\sqrt{2}}{3}$$

33. $D = \{(r, \theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$, so

$\iint_D e^{(x^2+y^2)^2}\, dA = \int_0^{2\pi} \int_0^1 e^{(r^2)^2}\, r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 re^{r^4}\,dr = 2\pi \int_0^1 re^{r^4}\,dr$. Using a calculator, we estimate $2\pi \int_0^1 re^{r^4}\,dr \approx 4.5951$.