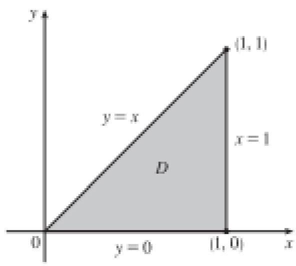
Section 15.3 - 5, 9, 13, 16, 19, 32, 37, 44, 51, 55, 57

5. $\int_0^1 \int_0^{s^2} \cos(s^3)\,dt\,ds = \int_0^1 \left[t\cos(s^3)\right]_{t=0}^{t=s^2} ds = \int_0^1 s^2\cos(s^3)\,ds = \frac{1}{3}\sin(s^3)\big]_0^1 = \frac{1}{3}(\sin 1 - \sin 0) = \frac{1}{3}\sin 1$

9. $\iint_D x\,dA = \int_0^\pi \int_0^{\sin x} x\,dy\,dx = \int_0^\pi [xy]_{y=0}^{y=\sin x}\,dx = \int_0^\pi x\sin x\,dx$ $\left[\begin{array}{c}\text{integrate by parts}\\ \text{with } u = x,\ dv = \sin x\,dx\end{array}\right]$

$= \left[-x\cos x + \sin x\right]_0^\pi = -\pi\cos\pi + \sin\pi + 0 - \sin 0 = \pi$

13.

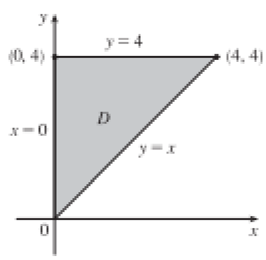


As a type I region, D lies between the lower boundary $y = 0$ and the upper boundary $y = x$ for $0 \le x \le 1$, so $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le x\}$. If we describe D as a type II region, D lies between the left boundary $x = y$ and the right boundary $x = 1$ for $0 \le y \le 1$, so $D = \{(x, y) \mid 0 \le y \le 1, y \le x \le 1\}$.

Thus $\iint_D x\,dA = \int_0^1\int_0^x x\,dy\,dx = \int_0^1 [xy]_{y=0}^{y=x}\,dx = \int_0^1 x^2\,dx = \frac{1}{3}x^3\big]_0^1 = \frac{1}{3}(1-0) = \frac{1}{3}$ or

$\iint_D x\,dA = \int_0^1\int_y^1 x\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2\right]_{x=y}^{x=1} dy = \frac{1}{2}\int_0^1 (1-y^2)\,dy = \frac{1}{2}\left[y - \frac{1}{3}y^3\right]_0^1 = \frac{1}{2}\left[\left(1-\frac{1}{3}\right) - 0\right] = \frac{1}{3}$.

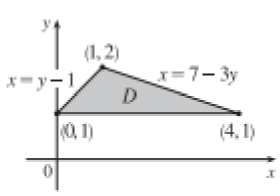
16.



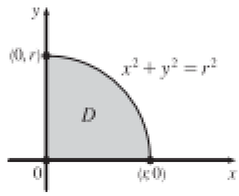
As a type I region, $D = \{(x, y) \mid 0 \le x \le 4, x \le y \le 4\}$ and $\iint_D y^2 e^{xy}\,dA = \int_0^4\int_x^4 y^2 e^{xy}\,dy\,dx$. As a type II region, $D = \{(x, y) \mid 0 \le y \le 4, 0 \le x \le y\}$ and $\iint_D y^2e^{xy}\,dA = \int_0^4\int_0^y y^2 e^{xy}\,dx\,dy$. Evaluating $\int y^2 e^{xy}\,dy$ requires integration by parts whereas $\int y^2 e^{xy}\,dx$ does not, so the iterated integral corresponding to D as a type II region appears easier to evaluate.

$$\begin{aligned}\iint_D y^2e^{xy}\,dA &= \int_0^4\int_0^y y^2e^{xy}\,dx\,dy = \int_0^4 \left[ye^{xy}\right]_{x=0}^{x=y} dy = \int_0^4 \left(ye^{y^2} - y\right) dy \\ &= \left[\tfrac{1}{2}e^{y^2} - \tfrac{1}{2}y^2\right]_0^4 = \left(\tfrac{1}{2}e^{16} - 8\right) - \left(\tfrac{1}{2} - 0\right) = \tfrac{1}{2}e^{16} - \tfrac{17}{2}\end{aligned}$$

19.



$$\begin{aligned}\iint_D y^2\,dA &= \int_1^2\int_{y-1}^{7-3y} y^2\,dx\,dy = \int_1^2 \left[xy^2\right]_{x=y-1}^{x=7-3y} dy \\ &= \int_1^2 \left[(7-3y)-(y-1)\right]y^2\,dy = \int_1^2 (8y^2 - 4y^3)\,dy \\ &= \left[\tfrac{8}{3}y^3 - y^4\right]_1^2 = \tfrac{64}{3} - 16 - \tfrac{8}{3} + 1 = \tfrac{11}{3}\end{aligned}$$

32. 

By symmetry, the desired volume V is 8 times the volume V_1 in the first octant.

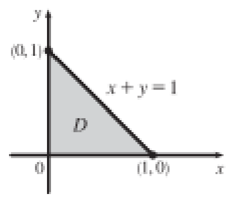
Now

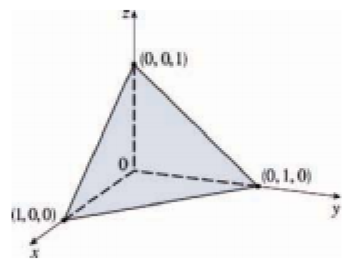
$$V_1 = \int_0^r \int_0^{\sqrt{r^2-y^2}} \sqrt{r^2-y^2}\,dx\,dy = \int_0^r \left[x\sqrt{r^2-y^2}\right]_{x=0}^{x=\sqrt{r^2-y^2}} dy$$

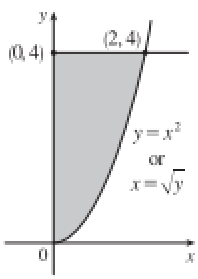
$$= \int_0^r (r^2-y^2)\,dy = \left[r^2y - \tfrac{1}{3}y^3\right]_0^r = \tfrac{2}{3}r^3$$

Thus $V = \frac{16}{3}r^3$.

37. The solid lies below the plane $z = 1 - x - y$ or $x + y + z = 1$ and above the region $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$ in the xy-plane. The solid is a tetrahedron.



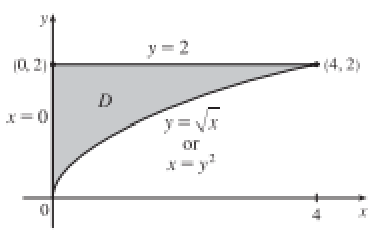


44. 

Because the region of integration is

$$D = \{(x,y) \mid x^2 \le y \le 4, 0 \le x \le 2\}$$
$$= \{(x,y) \mid 0 \le x \le \sqrt{y}, 0 \le y \le 4\}$$

we have $\int_0^2 \int_{x^2}^4 f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA = \int_0^4 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy$.

51. 

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx = \int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy$$

$$= \int_0^2 \frac{1}{y^3+1}\left[x\right]_{x=0}^{x=y^2} dy = \int_0^2 \frac{y^2}{y^3+1}\,dy$$

$$= \tfrac{1}{3}\ln\left|y^3+1\right|\Big]_0^2 = \tfrac{1}{3}(\ln 9 - \ln 1) = \tfrac{1}{3}\ln 9$$

55. $D = \{(x,y) \mid 0 \le x \le 1, -x+1 \le y \le 1\} \cup \{(x,y) \mid -1 \le x \le 0, x+1 \le y \le 1\}$
$\cup \{(x,y) \mid 0 \le x \le 1, -1 \le y \le x-1\} \cup \{(x,y) \mid -1 \le x \le 0, -1 \le y \le -x-1\}$, all type I.

$$\iint_D x^2\,dA = \int_0^1 \int_{1-x}^1 x^2\,dy\,dx + \int_{-1}^0 \int_{x+1}^1 x^2\,dy\,dx + \int_0^1 \int_{-1}^{x-1} x^2\,dy\,dx + \int_{-1}^0 \int_{-1}^{-x-1} x^2\,dy\,dx$$

$$= 4\int_0^1 \int_{1-x}^1 x^2\,dy\,dx \qquad \text{[by symmetry of the regions and because } f(x,y) = x^2 \ge 0\text{]}$$

$$= 4\int_0^1 x^3\,dx = 4\left[\tfrac{1}{4}x^4\right]_0^1 = 1$$

57. Here $Q = \left\{(x, y) \mid x^2 + y^2 \le \frac{1}{4}, x \ge 0, y \ge 0\right\}$, and $0 \le (x^2 + y^2)^2 \le \left(\frac{1}{4}\right)^2 \quad \Rightarrow \quad -\frac{1}{16} \le -(x^2 + y^2)^2 \le 0$ so $e^{-1/16} \le e^{-(x^2+y^2)^2} \le e^0 = 1$ since e^t is an increasing function. We have $A(Q) = \frac{1}{4}\pi \left(\frac{1}{2}\right)^2 = \frac{\pi}{16}$, so by Property 11, $e^{-1/16}\, A(Q) \le \iint_Q e^{-(x^2+y^2)^2} dA \le 1 \cdot A(Q) \quad \Rightarrow \quad \frac{\pi}{16} e^{-1/16} \le \iint_Q e^{-(x^2+y^2)^2} dA \le \frac{\pi}{16}$ or we can say $0.1844 < \iint_Q e^{-(x^2+y^2)^2} dA < 0.1964$. (We have rounded the lower bound down and the upper bound up to preserve the inequalities.)