3. $x = e^{-r}\sin\theta,\ y = e^{r}\cos\theta.$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} -e^{-r}\sin\theta & e^{-r}\cos\theta \\ e^{r}\cos\theta & -e^{r}\sin\theta \end{vmatrix} = e^{-r}e^{r}\sin^2\theta - e^{-r}e^{r}\cos^2\theta = \sin^2\theta - \cos^2\theta \text{ or } -\cos 2\theta$$

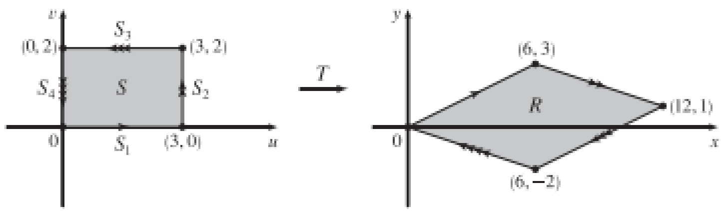
5. $x = u/v,\ y = v/w,\ z = w/u.$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} 1/v & -u/v^2 & 0 \\ 0 & 1/w & -v/w^2 \\ -w/u^2 & 0 & 1/u \end{vmatrix}$$

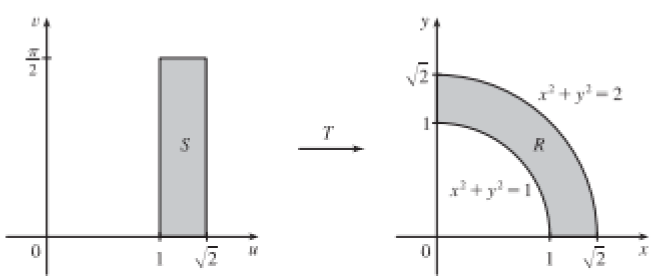
$$= \frac{1}{v}\begin{vmatrix} 1/w & -v/w^2 \\ 0 & 1/u \end{vmatrix} - \left(-\frac{u}{v^2}\right)\begin{vmatrix} 0 & -v/w^2 \\ -w/u^2 & 1/u \end{vmatrix} + 0\begin{vmatrix} 0 & 1/w \\ -w/u^2 & 0 \end{vmatrix}$$

$$= \frac{1}{v}\left(\frac{1}{uw} - 0\right) + \frac{u}{v^2}\left(0 - \frac{v}{u^2w}\right) + 0 = \frac{1}{uvw} - \frac{1}{uvw} = 0$$

7. The transformation maps the boundary of S to the boundary of the image R, so we first look at side S_1 in the uv-plane. S_1 is described by $v = 0, 0 \le u \le 3$, so $x = 2u + 3v = 2u$ and $y = u - v = u$. Eliminating u, we have $x = 2y, 0 \le x \le 6$. S_2 is the line segment $u = 3, 0 \le v \le 2$, so $x = 6 + 3v$ and $y = 3 - v$. Then $v = 3 - y \quad\Rightarrow\quad x = 6 + 3(3 - y) = 15 - 3y$, $6 \le x \le 12$. S_3 is the line segment $v = 2, 0 \le u \le 3$, so $x = 2u + 6$ and $y = u - 2$, giving $u = y + 2 \quad\Rightarrow\quad x = 2y + 10$, $6 \le x \le 12$. Finally, S_4 is the segment $u = 0, 0 \le v \le 2$, so $x = 3v$ and $y = -v \quad\Rightarrow\quad x = -3y, 0 \le x \le 6$. The image of set S is the region R shown in the xy-plane, a parallelogram bounded by these four segments.



13. R is a portion of an annular region (see the figure) that is easily described in polar coordinates as $R = \{(r,\theta) \mid 1 \le r \le \sqrt{2}, 0 \le \theta \le \pi/2\}$. If we converted a double integral over R to polar coordinates the resulting region of integration is a rectangle (in the $r\theta$-plane), so we can create a transformation T here by letting u play the role of r and v the role of θ. Thus T is defined by $x = u\cos v, y = u\sin v$ and T maps the rectangle $S = \{(u,v) \mid 1 \le u \le \sqrt{2}, 0 \le v \le \pi/2\}$ in the uv-plane to R in the xy-plane.



15. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ and $x - 3y = (2u + v) - 3(u + 2v) = -u - 5v$. To find the region S in the uv-plane that corresponds to R we first find the corresponding boundary under the given transformation. The line through $(0,0)$ and $(2,1)$ is $y = \frac{1}{2}x$ which is the image of $u + 2v = \frac{1}{2}(2u + v) \quad\Rightarrow\quad v = 0$; the line through $(2,1)$ and $(1,2)$ is $x + y = 3$ which is the image of $(2u + v) + (u + 2v) = 3 \quad\Rightarrow\quad u + v = 1$; the line through $(0,0)$ and $(1,2)$ is $y = 2x$ which is the image of $u + 2v = 2(2u + v) \quad\Rightarrow\quad u = 0$. Thus S is the triangle $0 \le v \le 1 - u$, $0 \le u \le 1$ in the uv-plane and

$$\iint_R (x - 3y)\,dA = \int_0^1 \int_0^{1-u} (-u - 5v)\,|3|\;dv\,du = -3\int_0^1 \left[uv + \tfrac{5}{2}v^2\right]_{v=0}^{v=1-u} du$$

$$= -3\int_0^1 \left(u - u^2 + \tfrac{5}{2}(1-u)^2\right) du = -3\left[\tfrac{1}{2}u^2 - \tfrac{1}{3}u^3 - \tfrac{5}{6}(1-u)^3\right]_0^1 = -3\left(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{5}{6}\right) = -3$$

19. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$, $xy = u$, $y = x$ is the image of the parabola $v^2 = u$, $y = 3x$ is the image of the parabola $v^2 = 3u$, and the hyperbolas $xy = 1$, $xy = 3$ are the images of the lines $u = 1$ and $u = 3$ respectively. Thus

$$\iint_R xy\,dA = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u\left(\frac{1}{v}\right) dv\,du = \int_1^3 u\left(\ln\sqrt{3u} - \ln\sqrt{u}\right) du = \int_1^3 u \ln\sqrt{3}\,du = 4\ln\sqrt{3} = 2\ln 3.$$

23. Letting $u = x - 2y$ and $v = 3x - y$, we have $x = \frac{1}{5}(2v - u)$ and $y = \frac{1}{5}(v - 3u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{vmatrix} = \dfrac{1}{5}$ and R is the image of the rectangle enclosed by the lines $u = 0$, $u = 4$, $v = 1$, and $v = 8$. Thus

$$\iint_R \frac{x - 2y}{3x - y}\,dA = \int_0^4 \int_1^8 \frac{u}{v}\left|\frac{1}{5}\right| dv\;du = \frac{1}{5}\int_0^4 u\,du \int_1^8 \frac{1}{v}\,dv = \tfrac{1}{5}\left[\tfrac{1}{2}u^2\right]_0^4 \left[\ln|v|\right]_1^8 = \tfrac{8}{5}\ln 8.$$