Section 14.8 - 3, 6, 9, 21, 27, 30, 37

3. $f(x,y) = x^2 + y^2$, $g(x,y) = xy = 1$, and $\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2y\rangle = \langle \lambda y, \lambda x\rangle$, so $2x = \lambda y$, $2y = \lambda x$, and $xy = 1$. From the last equation, $x \neq 0$ and $y \neq 0$, so $2x = \lambda y \Rightarrow \lambda = 2x/y$. Substituting, we have $2y = (2x/y)\,x \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$. But $xy = 1$, so $x = y = \pm 1$ and the possible points for the extreme values of f are $(1,1)$ and $(-1,-1)$. Here there is no maximum value, since the constraint $xy = 1$ allows x or y to become arbitrarily large, and hence $f(x,y) = x^2 + y^2$ can be made arbitrarily large. The minimum value is $f(1,1) = f(-1,-1) = 2$.

6. $f(x,y) = e^{xy}$, $g(x,y) = x^3 + y^3 = 16$, and $\nabla f = \lambda \nabla g \Rightarrow \langle ye^{xy}, xe^{xy}\rangle = \langle 3\lambda x^2, 3\lambda y^2\rangle$, so $ye^{xy} = 3\lambda x^2$ and $xe^{xy} = 3\lambda y^2$. Note that $x = 0 \Leftrightarrow y = 0$ which contradicts $x^3 + y^3 = 16$, so we may assume $x \neq 0$, $y \neq 0$, and then $\lambda = ye^{xy}/(3x^2) = xe^{xy}/(3y^2) \Rightarrow x^3 = y^3 \Rightarrow x = y$. But $x^3 + y^3 = 16$, so $2x^3 = 16 \Rightarrow x = 2 = y$.
Here there is no minimum value, since we can choose points satisfying the constraint $x^3 + y^3 = 16$ that make $f(x,y) = e^{xy}$ arbitrarily close to 0 (but never equal to 0). The maximum value is $f(2,2) = e^4$.

9. $f(x,y,z) = xyz$, $g(x,y,z) = x^2 + 2y^2 + 3z^2 = 6$. $\nabla f = \lambda \nabla g \Rightarrow \langle yz, xz, xy\rangle = \lambda\langle 2x, 4y, 6z\rangle$. If any of x, y, or z is zero then $x = y = z = 0$ which contradicts $x^2 + 2y^2 + 3z^2 = 6$. Then $\lambda = (yz)/(2x) = (xz)/(4y) = (xy)/(6z)$ or $x^2 = 2y^2$ and $z^2 = \frac{2}{3}y^2$. Thus $x^2 + 2y^2 + 3z^2 = 6$ implies $6y^2 = 6$ or $y = \pm 1$. Then the possible points are $\left(\sqrt{2}, \pm 1, \sqrt{\tfrac{2}{3}}\right)$, $\left(\sqrt{2}, \pm 1, -\sqrt{\tfrac{2}{3}}\right)$, $\left(-\sqrt{2}, \pm 1, \sqrt{\tfrac{2}{3}}\right)$, $\left(-\sqrt{2}, \pm 1, -\sqrt{\tfrac{2}{3}}\right)$. The maximum value of f on the ellipsoid is $\frac{2}{\sqrt{3}}$, occurring when all coordinates are positive or exactly two are negative and the minimum is $-\frac{2}{\sqrt{3}}$ occurring when 1 or 3 of the coordinates are negative.

21. $f(x,y) = e^{-xy}$. For the interior of the region, we find the critical points: $f_x = -ye^{-xy}$, $f_y = -xe^{-xy}$, so the only critical point is $(0,0)$, and $f(0,0) = 1$. For the boundary, we use Lagrange multipliers. $g(x,y) = x^2 + 4y^2 = 1 \Rightarrow \lambda\nabla g = \langle 2\lambda x, 8\lambda y\rangle$, so setting $\nabla f = \lambda \nabla g$ we get $-ye^{-xy} = 2\lambda x$ and $-xe^{-xy} = 8\lambda y$. The first of these gives $e^{-xy} = -2\lambda x/y$, and then the second gives $-x(-2\lambda x/y) = 8\lambda y \Rightarrow x^2 = 4y^2$. Solving this last equation with the constraint $x^2 + 4y^2 = 1$ gives $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{2\sqrt{2}}$. Now $f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{1/4} \approx 1.284$ and $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-1/4} \approx 0.779$. The former are the maxima on the region and the latter are the minima.

27. Let the sides of the rectangle be x and y. Then $f(x,y) = xy$, $g(x,y) = 2x + 2y = p \Rightarrow \nabla f(x,y) = \langle y, x\rangle$, $\lambda\nabla g = \langle 2\lambda, 2\lambda\rangle$. Then $\lambda = \frac{1}{2}y = \frac{1}{2}x$ implies $x = y$ and the rectangle with maximum area is a square with side length $\frac{1}{4}p$.

30. The distance from $(0,1,1)$ to a point (x,y,z) on the plane is $d = \sqrt{x^2 + (y-1)^2 + (z-1)^2}$, so we minimize $d^2 = f(x,y,z) = x^2 + (y-1)^2 + (z-1)^2$ subject to the constraint that (x,y,z) lies on the plane $x - 2y + 3z = 6$, that is, $g(x,y,z) = x - 2y + 3z = 6$. Then $\nabla f = \lambda\nabla g \Rightarrow \langle 2x, 2(y-1), 2(z-1)\rangle = \langle \lambda, -2\lambda, 3\lambda\rangle$, so $x = \lambda/2$, $y = 1 - \lambda$, $z = (3\lambda + 2)/2$. Substituting into the constraint equation gives $\dfrac{\lambda}{2} - 2(1-\lambda) + 3 \cdot \dfrac{3\lambda + 2}{2} = 6 \Rightarrow \lambda = \frac{5}{7}$, so $x = \frac{5}{14}$, $y = \frac{2}{7}$, and $z = \frac{29}{14}$. This must correspond to a minimum, so the point on the plane closest to the point $(0,1,1)$ is $\left(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}\right)$.

37. $f(x,y,z) = xyz,\ g(x,y,z) = x + 2y + 3z = 6 \quad \Rightarrow \quad \nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g = \langle \lambda, 2\lambda, 3\lambda \rangle$.

Then $\lambda = yz = \frac{1}{2}xz = \frac{1}{3}xy$ implies $x = 2y$, $z = \frac{2}{3}y$. But $2y + 2y + 2y = 6$ so $y = 1$, $x = 2$, $z = \frac{2}{3}$ and the volume is $V = \frac{4}{3}$.