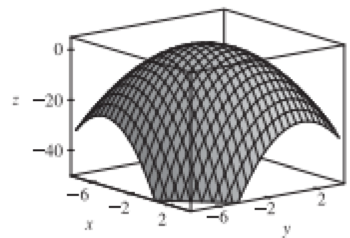
Section 14.7 - 1, 6, 13, 18, 29, 33, 40, 43, 51

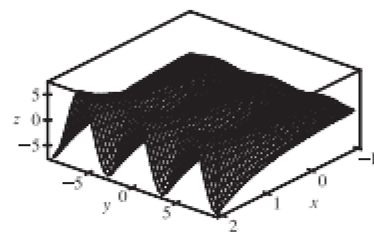
1. (a) First we compute $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (1)^2 = 7$. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$ by the Second Derivatives Test.

(b) $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (3)^2 = -1$. Since $D(1,1) < 0$, f has a saddle point at $(1,1)$ by the Second Derivatives Test.

6. $f(x,y) = xy - 2x - 2y - x^2 - y^2 \;\Rightarrow\; f_x = y - 2 - 2x$, $f_y = x - 2 - 2y$, $f_{xx} = -2$, $f_{xy} = 1$, $f_{yy} = -2$. Then $f_x = 0$ implies $y = 2x + 2$, and substitution into $f_y = 0$ gives $x - 2 - 2(2x+2) = 0 \;\Rightarrow\; -3x = 6 \;\Rightarrow\; x = -2$. Then $y = -2$ and the only critical point is $(-2,-2)$. $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-2) - 1^2 = 3$, and since $D(-2,-2) = 3 > 0$ and $f_{xx}(-2,-2) = -2 < 0$, $f(-2,-2) = 4$ is a local maximum by the Second Derivatives Test.

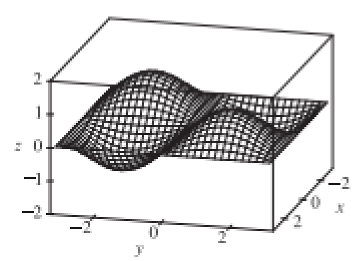


13. $f(x,y) = e^x \cos y \;\Rightarrow\; f_x = e^x \cos y$, $f_y = -e^x \sin y$. Now $f_x = 0$ implies $\cos y = 0$ or $y = \frac{\pi}{2} + n\pi$ for n an integer. But $\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$, so there are no critical points.



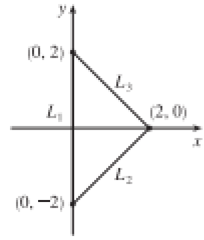
18. $f(x,y) = \sin x \sin y \;\Rightarrow\; f_x = \cos x \sin y$, $f_y = \sin x \cos y$, $f_{xx} = -\sin x \sin y$, $f_{xy} = \cos x \cos y$, $f_{yy} = -\sin x \sin y$. Here we have $-\pi < x < \pi$ and $-\pi < y < \pi$, so $f_x = 0$ implies $\cos x = 0$ or $\sin y = 0$. If $\cos x = 0$ then $x = -\frac{\pi}{2}$ or $\frac{\pi}{2}$, and if $\sin y = 0$ then $y = 0$. Substituting $x = \pm\frac{\pi}{2}$ into $f_y = 0$ gives $\cos y = 0 \;\Rightarrow\; y = -\frac{\pi}{2}$ or $\frac{\pi}{2}$, and substituting $y = 0$ into $f_y = 0$ gives $\sin x = 0 \;\Rightarrow\; x = 0$. Thus the critical points are $\left(-\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$, and $(0,0)$.

$D(0,0) = -1 < 0$ so $(0,0)$ is a saddle point. $D\left(-\frac{\pi}{2}, \pm\frac{\pi}{2}\right) = D\left(\frac{\pi}{2}, \pm\frac{\pi}{2}\right) = 1 > 0$ and $f_{xx}\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right) = f_{xx}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = -1 < 0$ while $f_{xx}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = f_{xx}\left(\frac{\pi}{2}, -\frac{\pi}{2}\right) = 1 > 0$, so $f\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = 1$ are local maxima and $f\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = f\left(\frac{\pi}{2}, -\frac{\pi}{2}\right) = 1$ are local minima.

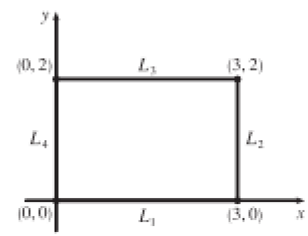


29. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. Here $f_x = 2x - 2$, $f_y = 2y$, and setting $f_x = f_y = 0$ gives $(1, 0)$ as the only critical point (which is inside D), where $f(1, 0) = -1$. Along L_1: $x = 0$ and $f(0, y) = y^2$ for $-2 \le y \le 2$, a quadratic function which attains its minimum at $y = 0$, where $f(0, 0) = 0$, and its maximum at $y = \pm 2$, where $f(0, \pm 2) = 4$. Along L_2: $y = x - 2$ for $0 \le x \le 2$, and $f(x, x - 2) = 2x^2 - 6x + 4 = 2\left(x - \frac{3}{2}\right)^2 - \frac{1}{2}$, a quadratic which attains its minimum at $x = \frac{3}{2}$, where $f\left(\frac{3}{2}, -\frac{1}{2}\right) = -\frac{1}{2}$, and its maximum at $x = 0$, where $f(0, -2) = 4$.

Along L_3: $y = 2 - x$ for $0 \le x \le 2$, and $f(x, 2 - x) = 2x^2 - 6x + 4 = 2\left(x - \frac{3}{2}\right)^2 - \frac{1}{2}$, a quadratic which attains its minimum at $x = \frac{3}{2}$, where $f\left(\frac{3}{2}, \frac{1}{2}\right) = -\frac{1}{2}$, and its maximum at $x = 0$, where $f(0, 2) = 4$. Thus the absolute maximum of f on D is $f(0, \pm 2) = 4$ and the absolute minimum is $f(1, 0) = -1$.



33. $f(x, y) = x^4 + y^4 - 4xy + 2$ is a polynomial and hence continuous on D, so it has an absolute maximum and minimum on D. $f_x(x, y) = 4x^3 - 4y$ and $f_y(x, y) = 4y^3 - 4x$; then $f_x = 0$ implies $y = x^3$, and substitution into $f_y = 0 \;\Rightarrow\; x = y^3$ gives $x^9 - x = 0 \;\Rightarrow\; x(x^8 - 1) = 0 \;\Rightarrow\; x = 0$ or $x = \pm 1$. Thus the critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$, but only $(1, 1)$ with $f(1, 1) = 0$ is inside D. On L_1: $y = 0$, $f(x, 0) = x^4 + 2$, $0 \le x \le 3$, a polynomial in x which attains its maximum at $x = 3$, $f(3, 0) = 83$, and its minimum at $x = 0$, $f(0, 0) = 2$. On L_2: $x = 3$, $f(3, y) = y^4 - 12y + 83$, $0 \le y \le 2$, a polynomial in y which attains its minimum at $y = \sqrt[3]{3}$, $f\left(3, \sqrt[3]{3}\right) = 83 - 9\sqrt[3]{3} \approx 70.0$, and its maximum at $y = 0$, $f(3, 0) = 83$.



On L_3: $y = 2$, $f(x, 2) = x^4 - 8x + 18$, $0 \le x \le 3$, a polynomial in x which attains its minimum at $x = \sqrt[3]{2}$, $f\left(\sqrt[3]{2}, 2\right) = 18 - 6\sqrt[3]{2} \approx 10.4$, and its maximum at $x = 3$, $f(3, 2) = 75$. On L_4: $x = 0$, $f(0, y) = y^4 + 2$, $0 \le y \le 2$, a polynomial in y which attains its maximum at $y = 2$, $f(0, 2) = 18$, and its minimum at $y = 0$, $f(0, 0) = 2$. Thus the absolute maximum of f on D is $f(3, 0) = 83$ and the absolute minimum is $f(1, 1) = 0$.

40. Here the distance d from a point on the plane to the point $(0, 1, 1)$ is $d = \sqrt{x^2 + (y - 1)^2 + (z - 1)^2}$, where $z = 2 - \frac{1}{3}x + \frac{2}{3}y$. We can minimize $d^2 = f(x, y) = x^2 + (y - 1)^2 + \left(1 - \frac{1}{3}x + \frac{2}{3}y\right)^2$, so

$f_x(x, y) = 2x + 2\left(1 - \frac{1}{3}x + \frac{2}{3}y\right)\left(-\frac{1}{3}\right) = \frac{20}{9}x - \frac{4}{9}y - \frac{2}{3}$ and

$f_y(x, y) = 2(y - 1) + 2\left(1 - \frac{1}{3}x + \frac{2}{3}y\right)\left(\frac{2}{3}\right) = -\frac{4}{9}x + \frac{26}{9}y - \frac{2}{3}$. Solving $\frac{20}{9}x - \frac{4}{9}y - \frac{2}{3} = 0$ and $-\frac{4}{9}x + \frac{26}{9}y - \frac{2}{3} = 0$ simultaneously gives $x = \frac{5}{14}$ and $y = \frac{2}{7}$, so the only critical point is $\left(\frac{5}{14}, \frac{2}{7}\right)$.

This point must correspond to the minimum distance, so the point on the plane closest to $(0, 1, 1)$ is $\left(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}\right)$.

43. $x + y + z = 100$, so maximize $f(x, y) = xy(100 - x - y)$. $f_x = 100y - 2xy - y^2$, $f_y = 100x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 100 - 2x - 2y$. Then $f_x = 0$ implies $y = 0$ or $y = 100 - 2x$. Substituting $y = 0$ into $f_y = 0$ gives $x = 0$ or $x = 100$ and substituting $y = 100 - 2x$ into $f_y = 0$ gives $3x^2 - 100x = 0$ so $x = 0$ or $\frac{100}{3}$.

Thus the critical points are $(0, 0)$, $(100, 0)$, $(0, 100)$ and $\left(\frac{100}{3}, \frac{100}{3}\right)$.

$D(0, 0) = D(100, 0) = D(0, 100) = -10{,}000$ while $D\left(\frac{100}{3}, \frac{100}{3}\right) = \frac{10{,}000}{3}$ and $f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) = -\frac{200}{3} < 0$. Thus $(0, 0)$, $(100, 0)$ and $(0, 100)$ are saddle points whereas $f\left(\frac{100}{3}, \frac{100}{3}\right)$ is a local maximum. Thus the numbers are $x = y = z = \frac{100}{3}$.

51. Let the dimensions be x, y and z, then minimize $xy + 2(xz + yz)$ if $xyz = 32{,}000\ \text{cm}^3$. Then $f(x, y) = xy + [64{,}000(x + y)/xy] = xy + 64{,}000(x^{-1} + y^{-1})$, $f_x = y - 64{,}000x^{-2}$, $f_y = x - 64{,}000y^{-2}$. And $f_x = 0$ implies $y = 64{,}000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64{,}000$ or $x = 40$ and then $y = 40$. Now $D(x, y) = [(2)(64{,}000)]^2 x^{-3} y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40, 40) > 0$ so this is indeed a minimum. Thus the dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.