Section 14.6 - 5, 10, 13, 17, 20, 23, 29, 41, 46, 49

5. $f(x,y) = ye^{-x} \Rightarrow f_x(x,y) = -ye^{-x}$ and $f_y(x,y) = e^{-x}$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = 2\pi/3$, then from Equation 6, $D_\mathbf{u} f(0,4) = f_x(0,4)\cos\left(\frac{2\pi}{3}\right) + f_y(0,4)\sin\left(\frac{2\pi}{3}\right) = -4 \cdot \left(-\frac{1}{2}\right) + 1 \cdot \frac{\sqrt{3}}{2} = 2 + \frac{\sqrt{3}}{2}$.

10. $f(x,y,z) = y^2 e^{xyz}$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \langle y^2 e^{xyz}(yz), y^2 \cdot e^{xyz}(xz) + e^{xyz} \cdot 2y, y^2 e^{xyz}(xy)\rangle$
$= \langle y^3 z e^{xyz}, (xy^2 z + 2y)e^{xyz}, xy^3 e^{xyz}\rangle$

(b) $\nabla f(0,1,-1) = \langle -1, 2, 0\rangle$

(c) $D_\mathbf{u} f(0,1,-1) = \nabla f(0,1,-1) \cdot \mathbf{u} = \langle -1,2,0\rangle \cdot \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right\rangle = -\frac{3}{13} + \frac{8}{13} + 0 = \frac{5}{13}$

13. $g(p,q) = p^4 - p^2 q^3 \Rightarrow \nabla g(p,q) = (4p^3 - 2pq^3)\,\mathbf{i} + (-3p^2q^2)\,\mathbf{j}$, $\nabla g(2,1) = 28\,\mathbf{i} - 12\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{1^2+3^2}}(\mathbf{i} + 3\,\mathbf{j}) = \frac{1}{\sqrt{10}}(\mathbf{i} + 3\,\mathbf{j})$, so

$D_\mathbf{u}\, g(2,1) = \nabla g(2,1) \cdot \mathbf{u} = (28\,\mathbf{i} - 12\,\mathbf{j}) \cdot \frac{1}{\sqrt{10}}(\mathbf{i} + 3\,\mathbf{j}) = \frac{1}{\sqrt{10}}(28 - 36) = -\frac{8}{\sqrt{10}}$ or $-\frac{4\sqrt{10}}{5}$.

17. $h(r,s,t) = \ln(3r + 6s + 9t) \Rightarrow \nabla h(r,s,t) = \langle 3/(3r+6s+9t), 6/(3r+6s+9t), 9/(3r+6s+9t)\rangle$,

$\nabla h(1,1,1) = \left\langle \frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right\rangle$, and a unit vector in the direction of $\mathbf{v} = 4\,\mathbf{i} + 12\,\mathbf{j} + 6\,\mathbf{k}$

is $\mathbf{u} = \frac{1}{\sqrt{16+144+36}}(4\,\mathbf{i} + 12\,\mathbf{j} + 6\,\mathbf{k}) = \frac{2}{7}\,\mathbf{i} + \frac{6}{7}\,\mathbf{j} + \frac{3}{7}\,\mathbf{k}$, so

$D_\mathbf{u}\, h(1,1,1) = \nabla h(1,1,1) \cdot \mathbf{u} = \left\langle \frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right\rangle \cdot \left\langle \frac{2}{7}, \frac{6}{7}, \frac{3}{7}\right\rangle = \frac{1}{21} + \frac{2}{7} + \frac{3}{14} = \frac{23}{42}$.

20. $f(x,y,z) = xy + yz + zx \Rightarrow \nabla f(x,y,z) = \langle y+z, x+z, y+x\rangle$, so $\nabla f(1,-1,3) = \langle 2,4,0\rangle$. The unit vector in the direction of $\overrightarrow{PQ} = \langle 1,5,2\rangle$ is $\mathbf{u} = \frac{1}{\sqrt{30}}\langle 1,5,2\rangle$, so $D_\mathbf{u} f(1,-1,3) = \nabla f(1,-1,3) \cdot \mathbf{u} = \langle 2,4,0\rangle \cdot \frac{1}{\sqrt{30}}\langle 1,5,2\rangle = \frac{22}{\sqrt{30}}$.

23. $f(x,y) = \sin(xy) \Rightarrow \nabla f(x,y) = \langle y\cos(xy), x\cos(xy)\rangle$, $\nabla f(1,0) = \langle 0,1\rangle$. Thus the maximum rate of change is $|\nabla f(1,0)| = 1$ in the direction $\langle 0,1\rangle$.

29. The direction of fastest change is $\nabla f(x,y) = (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j}$, so we need to find all points (x,y) where $\nabla f(x,y)$ is parallel to $\mathbf{i} + \mathbf{j} \Leftrightarrow (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j} = k(\mathbf{i} + \mathbf{j}) \Leftrightarrow k = 2x - 2$ and $k = 2y - 4$. Then $2x - 2 = 2y - 4 \Rightarrow y = x + 1$, so the direction of fastest change is $\mathbf{i} + \mathbf{j}$ at all points on the line $y = x + 1$.

41. Let $F(x,y,z) = 2(x-2)^2 + (y-1)^2 + (z-3)^2$. Then $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10$ is a level surface of F.

$F_x(x,y,z) = 4(x-2) \Rightarrow F_x(3,3,5) = 4$, $F_y(x,y,z) = 2(y-1) \Rightarrow F_y(3,3,5) = 4$, and

$F_z(x,y,z) = 2(z-3) \Rightarrow F_z(3,3,5) = 4$.

(a) Equation 19 gives an equation of the tangent plane at $(3,3,5)$ as $4(x-3) + 4(y-3) + 4(z-5) = 0 \Leftrightarrow$
$4x + 4y + 4z = 44$ or equivalently $x + y + z = 11$.

46. Let $F(x, y, z) = x^4 + y^4 + z^4 - 3x^2y^2z^2$. Then $x^4 + y^4 + z^4 = 3x^2y^2z^2$ is the level surface $F(x, y, z) = 0$, and $\nabla F(x, y, z) = \langle 4x^3 - 6xy^2z^2, 4y^3 - 6x^2yz^2, 4z^3 - 6x^2y^2z \rangle$.

(a) $\nabla F(1, 1, 1) = \langle -2, -2, -2 \rangle$ or equivalently $\langle 1, 1, 1 \rangle$ is a normal vector for the tangent plane at $(1, 1, 1)$, so an equation of the tangent plane is $1(x - 1) + 1(y - 1) + 1(z - 1) = 0$ or $x + y + z = 3$.

(b) The normal line has direction $\langle 1, 1, 1 \rangle$, so parametric equations are $x = 1 + t,\ y = 1 + t,\ z = 1 + t$, and symmetric equations are $x - 1 = y - 1 = z - 1$ or equivalently $x = y = z$.

49. $f(x, y) = xy \Rightarrow \nabla f(x, y) = \langle y, x \rangle$, $\nabla f(3, 2) = \langle 2, 3 \rangle$. $\nabla f(3, 2)$ is perpendicular to the tangent line, so the tangent line has equation $\nabla f(3, 2) \cdot \langle x - 3, y - 2 \rangle = 0 \Rightarrow \langle 2, 3 \rangle \cdot \langle x - 3, x - 2 \rangle = 0 \Rightarrow$ $2(x - 3) + 3(y - 2) = 0$ or $2x + 3y = 12$.