Section 14.3 - 1, 3, 22, 24, 29, 43, 45, 50, 51, 55, 61, 65

1. (a) $\partial T/\partial x$ represents the rate of change of T when we fix y and t and consider T as a function of the single variable x, which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of T when we fix x and t and consider T as a function of y, which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of T when we fix x and y and consider T as a function of t, which describes how quickly the temperature changes over time for a constant longitude and latitude.

(b) $f_x(158, 21, 9)$ represents the rate of change of temperature at longitude $158°$W, latitude $21°$N at 9:00 AM when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158, 21, 9)$ to be positive. $f_y(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158, 21, 9)$ to be negative. $f_t(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only time varies. Since typically air temperature increases from the morning to the afternoon as the sun warms it, we would expect $f_t(158, 21, 9)$ to be positive.

3. (a) By Definition 4, $f_T(-15,30) = \lim\limits_{h\to 0} \dfrac{f(-15+h,30) - f(-15,30)}{h}$, which we can approximate by considering $h = 5$ and $h = -5$ and using the values given in the table:

$f_T(-15,30) \approx \dfrac{f(-10,30) - f(-15,30)}{5} = \dfrac{-20-(-26)}{5} = \dfrac{6}{5} = 1.2$,

$f_T(-15,30) \approx \dfrac{f(-20,30) - f(-15,30)}{-5} = \dfrac{-33-(-26)}{-5} = \dfrac{-7}{-5} = 1.4$. Averaging these values, we estimate $f_T(-15,30)$ to be approximately 1.3. Thus, when the actual temperature is $-15°C$ and the wind speed is 30 km/h, the apparent temperature rises by about $1.3°C$ for every degree that the actual temperature rises.

Similarly, $f_v(-15,30) = \lim\limits_{h\to 0} \dfrac{f(-15,30+h) - f(-15,30)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$: $f_v(-15,30) \approx \dfrac{f(-15,40) - f(-15,30)}{10} = \dfrac{-27-(-26)}{10} = \dfrac{-1}{10} = -0.1$,

$f_v(-15,30) \approx \dfrac{f(-15,20) - f(-15,30)}{-10} = \dfrac{-24-(-26)}{-10} = \dfrac{2}{-10} = -0.2$. Averaging these values, we estimate $f_v(-15,30)$ to be approximately -0.15. Thus, when the actual temperature is $-15°C$ and the wind speed is 30 km/h, the apparent temperature decreases by about $0.15°C$ for every km/h that the wind speed increases.

(b) For a fixed wind speed v, the values of the wind-chill index W increase as temperature T increases (look at a column of the table), so $\dfrac{\partial W}{\partial T}$ is positive. For a fixed temperature T, the values of W decrease (or remain constant) as v increases (look at a row of the table), so $\dfrac{\partial W}{\partial v}$ is negative (or perhaps 0).

(c) For fixed values of T, the function values $f(T,v)$ appear to become constant (or nearly constant) as v increases, so the corresponding rate of change is 0 or near 0 as v increases. This suggests that $\lim\limits_{v\to\infty} (\partial W/\partial v) = 0$.

22. $f(x,y) = \dfrac{x}{(x+y)^2} \quad\Rightarrow\quad f_x(x,y) = \dfrac{(x+y)^2(1) - (x)(2)(x+y)}{[(x+y)^2]^2} = \dfrac{x+y-2x}{(x+y)^3} = \dfrac{y-x}{(x+y)^3}$,

$f_y(x,y) = \dfrac{(x+y)^2(0) - (x)(2)(x+y)}{[(x+y)^2]^2} = -\dfrac{2x}{(x+y)^3}$

24. $w = \dfrac{e^v}{u+v^2} \quad\Rightarrow\quad \dfrac{\partial w}{\partial u} = \dfrac{0(u+v^2) - e^v(1)}{(u+v^2)^2} = -\dfrac{e^v}{(u+v^2)^2}$, $\dfrac{\partial w}{\partial v} = \dfrac{e^v(u+v^2) - e^v(2v)}{(u+v^2)^2} = \dfrac{e^v(u+v^2-2v)}{(u+v^2)^2}$

29. $F(x,y) = \displaystyle\int_y^x \cos(e^t)\,dt \quad\Rightarrow\quad F_x(x,y) = \dfrac{\partial}{\partial x}\int_y^x \cos(e^t)\,dt = \cos(e^x)$ by the Fundamental Theorem of Calculus, Part 1;

$F_y(x,y) = \dfrac{\partial}{\partial y}\displaystyle\int_y^x \cos(e^t)\,dt = \dfrac{\partial}{\partial y}\left[-\int_x^y \cos(e^t)\,dt\right] = -\dfrac{\partial}{\partial y}\int_x^y \cos(e^t)\,dt = -\cos(e^y)$.

43. $f(x,y,z) = \dfrac{y}{x+y+z} \quad\Rightarrow\quad f_y(x,y,z) = \dfrac{1(x+y+z) - y(1)}{(x+y+z)^2} = \dfrac{x+z}{(x+y+z)^2}$,

so $f_y(2,1,-1) = \dfrac{2+(-1)}{(2+1+(-1))^2} = \dfrac{1}{4}$.

45. $f(x,y) = xy^2 - x^3y \quad \Rightarrow$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \frac{(x+h)y^2 - (x+h)^3y - (xy^2 - x^3y)}{h}$$
$$= \lim_{h\to 0} \frac{h(y^2 - 3x^2y - 3xyh - yh^2)}{h} = \lim_{h\to 0}(y^2 - 3x^2y - 3xyh - yh^2) = y^2 - 3x^2y$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} = \lim_{h\to 0} \frac{x(y+h)^2 - x^3(y+h) - (xy^2 - x^3y)}{h} = \lim_{h\to 0} \frac{h(2xy + xh - x^3)}{h}$$
$$= \lim_{h\to 0}(2xy + xh - x^3) = 2xy - x^3$$

50. $yz + x\ln y = z^2 \quad \Rightarrow \quad \frac{\partial}{\partial x}(yz + x\ln y) = \frac{\partial}{\partial x}(z^2) \quad \Rightarrow \quad y\frac{\partial z}{\partial x} + \ln y = 2z\frac{\partial z}{\partial x} \quad \Rightarrow \quad \ln y = 2z\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial x} \quad \Rightarrow$

$\ln y = (2z - y)\frac{\partial z}{\partial x}$, so $\frac{\partial z}{\partial x} = \frac{\ln y}{2z - y}$.

$\frac{\partial}{\partial y}(yz + x\ln y) = \frac{\partial}{\partial y}(z^2) \quad \Rightarrow \quad y\frac{\partial z}{\partial y} + z \cdot 1 + x \cdot \frac{1}{y} = 2z\frac{\partial z}{\partial y} \quad \Rightarrow \quad z + \frac{x}{y} = 2z\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial y} \quad \Rightarrow$

$z + \frac{x}{y} = (2z - y)\frac{\partial z}{\partial y}$, so $\frac{\partial z}{\partial y} = \frac{z + (x/y)}{2z - y} = \frac{x + yz}{y(2z - y)}$.

51. (a) $z = f(x) + g(y) \quad \Rightarrow \quad \frac{\partial z}{\partial x} = f'(x), \quad \frac{\partial z}{\partial y} = g'(y)$

(b) $z = f(x+y)$. Let $u = x + y$. Then $\frac{\partial z}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = \frac{df}{du}(1) = f'(u) = f'(x+y)$,

$\frac{\partial z}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y} = \frac{df}{du}(1) = f'(u) = f'(x+y)$.

55. $w = \sqrt{u^2 + v^2} \quad \Rightarrow \quad w_u = \frac{1}{2}(u^2+v^2)^{-1/2} \cdot 2u = \frac{u}{\sqrt{u^2+v^2}}$, $w_v = \frac{1}{2}(u^2+v^2)^{-1/2} \cdot 2v = \frac{v}{\sqrt{u^2+v^2}}$. Then

$$w_{uu} = \frac{1 \cdot \sqrt{u^2+v^2} - u \cdot \frac{1}{2}(u^2+v^2)^{-1/2}(2u)}{\left(\sqrt{u^2+v^2}\right)^2} = \frac{\sqrt{u^2+v^2} - u^2/\sqrt{u^2+v^2}}{u^2+v^2} = \frac{u^2+v^2-u^2}{(u^2+v^2)^{3/2}} = \frac{v^2}{(u^2+v^2)^{3/2}},$$
$$w_{uv} = u\left(-\tfrac{1}{2}\right)\left(u^2+v^2\right)^{-3/2}(2v) = -\frac{uv}{(u^2+v^2)^{3/2}}, \quad w_{vu} = v\left(-\tfrac{1}{2}\right)\left(u^2+v^2\right)^{-3/2}(2u) = -\frac{uv}{(u^2+v^2)^{3/2}},$$
$$w_{vv} = \frac{1 \cdot \sqrt{u^2+v^2} - v \cdot \frac{1}{2}(u^2+v^2)^{-1/2}(2v)}{\left(\sqrt{u^2+v^2}\right)^2} = \frac{\sqrt{u^2+v^2} - v^2/\sqrt{u^2+v^2}}{u^2+v^2} = \frac{u^2+v^2-v^2}{(u^2+v^2)^{3/2}} = \frac{u^2}{(u^2+v^2)^{3/2}}.$$

61. $u = \cos(x^2y) \quad \Rightarrow \quad u_x = -\sin(x^2y) \cdot 2xy = -2xy\sin(x^2y)$,

$u_{xy} = -2xy \cdot \cos(x^2y) \cdot x^2 + \sin(x^2y) \cdot (-2x) = -2x^3y\cos(x^2y) - 2x\sin(x^2y)$ and

$u_y = -\sin(x^2y) \cdot x^2 = -x^2\sin(x^2y)$, $\quad u_{yx} = -x^2 \cdot \cos(x^2y) \cdot 2xy + \sin(x^2y) \cdot (-2x) = -2x^3y\cos(x^2y) - 2x\sin(x^2y)$.

Thus $u_{xy} = u_{yx}$.

65. $f(x,y,z) = e^{xyz^2} \quad \Rightarrow \quad f_x = e^{xyz^2} \cdot yz^2 = yz^2e^{xyz^2}$, $f_{xy} = yz^2 \cdot e^{xyz^2}(xz^2) + e^{xyz^2} \cdot z^2 = (xyz^4 + z^2)e^{xyz^2}$,

$f_{xyz} = (xyz^4 + z^2) \cdot e^{xyz^2}(2xyz) + e^{xyz^2} \cdot (4xyz^3 + 2z) = (2x^2y^2z^5 + 6xyz^3 + 2z)e^{xyz^2}$.