Section 13.4 - 1, 7, 13, 16, 19, 25, 26

1. (a) If $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is the position vector of the particle at time t, then the average velocity over the time interval $[0, 1]$ is

$\mathbf{v}_{\text{ave}} = \dfrac{\mathbf{r}(1) - \mathbf{r}(0)}{1 - 0} = \dfrac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (2.7\,\mathbf{i} + 9.8\,\mathbf{j} + 3.7\,\mathbf{k})}{1} = 1.8\,\mathbf{i} - 3.8\,\mathbf{j} - 0.7\,\mathbf{k}$. Similarly, over the other intervals we have

$$[0.5, 1]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1) - \mathbf{r}(0.5)}{1 - 0.5} = \frac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (3.5\,\mathbf{i} + 7.2\,\mathbf{j} + 3.3\,\mathbf{k})}{0.5} = 2.0\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}$$

$$[1, 2]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(2) - \mathbf{r}(1)}{2 - 1} = \frac{(7.3\,\mathbf{i} + 7.8\,\mathbf{j} + 2.7\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{1} = 2.8\,\mathbf{i} + 1.8\,\mathbf{j} - 0.3\,\mathbf{k}$$

$$[1, 1.5]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1.5) - \mathbf{r}(1)}{1.5 - 1} = \frac{(5.9\,\mathbf{i} + 6.4\,\mathbf{j} + 2.8\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{0.5} = 2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k}$$

(b) We can estimate the velocity at $t = 1$ by averaging the average velocities over the time intervals $[0.5, 1]$ and $[1, 1.5]$:

$\mathbf{v}(1) \approx \frac{1}{2}[(2\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}) + (2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k})] = 2.4\,\mathbf{i} - 0.8\,\mathbf{j} - 0.5\,\mathbf{k}$. Then the speed is

$|\mathbf{v}(1)| \approx \sqrt{(2.4)^2 + (-0.8)^2 + (-0.5)^2} \approx 2.58$.

7. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 2\,\mathbf{k} \quad \Rightarrow$

$\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j}$

$\mathbf{a}(t) = 2\,\mathbf{j}$

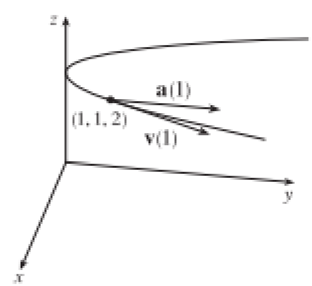
$|\mathbf{v}(t)| = \sqrt{1 + 4t^2}$

At $t = 1$:

$\mathbf{v}(1) = \mathbf{i} + 2\,\mathbf{j}$

$\mathbf{a}(1) = 2\,\mathbf{j}$

Here $x = t$, $y = t^2 \quad \Rightarrow \quad y = x^2$ and $z = 2$, so the path of the particle is a parabola in the plane $z = 2$.



13. $\mathbf{r}(t) = e^t \langle \cos t, \sin t, t \rangle \quad \Rightarrow$

$\mathbf{v}(t) = \mathbf{r}'(t) = e^t \langle \cos t, \sin t, t \rangle + e^t \langle -\sin t, \cos t, 1 \rangle = e^t \langle \cos t - \sin t, \sin t + \cos t, t + 1 \rangle$

$$\begin{aligned} \mathbf{a}(t) = \mathbf{v}'(t) &= e^t \langle \cos t - \sin t - \sin t - \cos t, \sin t + \cos t + \cos t - \sin t, t + 1 + 1 \rangle \\ &= e^t \langle -2\sin t, 2\cos t, t + 2 \rangle \end{aligned}$$

$$\begin{aligned} |\mathbf{v}(t)| &= e^t \sqrt{\cos^2 t + \sin^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\sin t \cos t + t^2 + 2t + 1} \\ &= e^t \sqrt{t^2 + 2t + 3} \end{aligned}$$

16. $\mathbf{a}(t) = 2\,\mathbf{i} + 6t\,\mathbf{j} + 12t^2\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \int (2\,\mathbf{i} + 6t\,\mathbf{j} + 12t^2\,\mathbf{k})\,dt = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + 4t^3\,\mathbf{k} + \mathbf{C}$, and $\mathbf{i} = \mathbf{v}(0) = \mathbf{C}$, so $\mathbf{C} = \mathbf{i}$ and $\mathbf{v}(t) = (2t + 1)\mathbf{i} + 3t^2\,\mathbf{j} + 4t^3\,\mathbf{k}$. $\quad \mathbf{r}(t) = \int \left[(2t + 1)\mathbf{i} + 3t^2\,\mathbf{j} + 4t^3\,\mathbf{k}\right] dt = (t^2 + t)\,\mathbf{i} + t^3\,\mathbf{j} + t^4\,\mathbf{k} + \mathbf{D}$. But $\mathbf{j} - \mathbf{k} = \mathbf{r}(0) = \mathbf{D}$, so $\mathbf{D} = \mathbf{j} - \mathbf{k}$ and $\mathbf{r}(t) = (t^2 + t)\,\mathbf{i} + (t^3 + 1)\,\mathbf{j} + (t^4 - 1)\,\mathbf{k}$.

19. $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle \Rightarrow \mathbf{v}(t) = \langle 2t, 5, 2t - 16 \rangle$, $|\mathbf{v}(t)| = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} = \sqrt{8t^2 - 64t + 281}$

and $\frac{d}{dt}|\mathbf{v}(t)| = \frac{1}{2}(8t^2 - 64t + 281)^{-1/2}(16t - 64)$. This is zero if and only if the numerator is zero, that is,

$16t - 64 = 0$ or $t = 4$. Since $\frac{d}{dt}|\mathbf{v}(t)| < 0$ for $t < 4$ and $\frac{d}{dt}|\mathbf{v}(t)| > 0$ for $t > 4$, the minimum speed of $\sqrt{153}$ is attained at $t = 4$ units of time.

25. As in Example 5, $\mathbf{r}(t) = (v_0 \cos 45°)t\,\mathbf{i} + \left[(v_0 \sin 45°)t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{1}{2}\left[v_0\sqrt{2}\,t\,\mathbf{i} + \left(v_0\sqrt{2}\,t - gt^2\right)\mathbf{j}\right]$. The ball lands when $y = 0$ (and $t > 0$) $\Rightarrow$ $t = \frac{v_0\sqrt{2}}{g}$ s. Now since it lands 90 m away, $90 = x = \frac{1}{2}v_0\sqrt{2}\,\frac{v_0\sqrt{2}}{g}$ or $v_0^2 = 90g$ and the initial velocity is $v_0 = \sqrt{90g} \approx 30$ m/s.

26. As in Example 5, $\mathbf{r}(t) = (v_0 \cos 30°)t\,\mathbf{i} + \left[(v_0 \sin 30°)t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{1}{2}\left[v_0\sqrt{3}\,t\,\mathbf{i} + (v_0\,t - gt^2)\,\mathbf{j}\right]$ and then $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{1}{2}\left[v_0\sqrt{3}\,\mathbf{i} + (v_0 - 2gt)\,\mathbf{j}\right]$. The shell reaches its maximum height when the vertical component of velocity is zero, so $\frac{1}{2}(v_0 - 2gt) = 0 \Rightarrow t = \frac{v_0}{2g}$. The vertical height of the shell at that time is 500 m,

so $\frac{1}{2}\left[v_0\left(\frac{v_0}{2g}\right) - g\left(\frac{v_0}{2g}\right)^2\right] = 500 \Rightarrow \frac{v_0^2}{8g} = 500 \Rightarrow v_0 = \sqrt{4000g} = \sqrt{4000(9.8)} \approx 198$ m/s.