Section 13.3 - 3, 5, 11, 14, 19, 21, 25, 29, 30

3. $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} \quad\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$ [since $e^t + e^{-t} > 0$].

Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^1 = e - e^{-1}$.

5. $\mathbf{r}(t) = \mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 2t\,\mathbf{j} + 3t^2\,\mathbf{k} \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{4t^2 + 9t^4} = t\,\sqrt{4 + 9t^2}$ [since $t \ge 0$].

Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 t\,\sqrt{4 + 9t^2}\,dt = \frac{1}{18}\cdot\frac{2}{3}(4 + 9t^2)^{3/2}\Big]_0^1 = \frac{1}{27}(13^{3/2} - 4^{3/2}) = \frac{1}{27}(13^{3/2} - 8)$.

11. The projection of the curve C onto the xy-plane is the curve $x^2 = 2y$ or $y = \frac{1}{2}x^2$, $z = 0$. Then we can choose the parameter $x = t \quad\Rightarrow\quad y = \frac{1}{2}t^2$. Since C also lies on the surface $3z = xy$, we have $z = \frac{1}{3}xy = \frac{1}{3}(t)(\frac{1}{2}t^2) = \frac{1}{6}t^3$. Then parametric equations for C are $x = t$, $y = \frac{1}{2}t^2$, $z = \frac{1}{6}t^3$ and the corresponding vector equation is $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, \frac{1}{6}t^3\rangle$. The origin corresponds to $t = 0$ and the point $(6, 18, 36)$ corresponds to $t = 6$, so

$$L = \int_0^6 |\mathbf{r}'(t)|\,dt = \int_0^6 \left|\langle 1, t, \tfrac{1}{2}t^2\rangle\right|\,dt = \int_0^6 \sqrt{1^2 + t^2 + \left(\tfrac{1}{2}t^2\right)^2}\,dt = \int_0^6 \sqrt{1 + t^2 + \tfrac{1}{4}t^4}\,dt$$
$$= \int_0^6 \sqrt{(1 + \tfrac{1}{2}t^2)^2}\,dt = \int_0^6 (1 + \tfrac{1}{2}t^2)\,dt = \left[t + \tfrac{1}{6}t^3\right]_0^6 = 6 + 36 = 42$$

14. $\mathbf{r}(t) = e^{2t}\cos 2t\,\mathbf{i} + 2\,\mathbf{j} + e^{2t}\sin 2t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 2e^{2t}(\cos 2t - \sin 2t)\,\mathbf{i} + 2e^{2t}(\cos 2t + \sin 2t)\,\mathbf{k}$,

$\frac{ds}{dt} = |\mathbf{r}'(t)| = 2e^{2t}\,\sqrt{(\cos 2t - \sin 2t)^2 + (\cos 2t + \sin 2t)^2} = 2e^{2t}\,\sqrt{2\cos^2 2t + 2\sin^2 2t} = 2\,\sqrt{2}\,e^{2t}$.

$s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 2\,\sqrt{2}\,e^{2u}\,du = \sqrt{2}\,e^{2u}\Big]_0^t = \sqrt{2}\,(e^{2t} - 1) \quad\Rightarrow\quad \frac{s}{\sqrt{2}} + 1 = e^{2t} \quad\Rightarrow\quad t = \frac{1}{2}\ln\left(\frac{s}{\sqrt{2}} + 1\right)$.

Substituting, we have

$$\mathbf{r}(t(s)) = e^{2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)}\cos 2\left(\tfrac{1}{2}\ln\left(\tfrac{s}{\sqrt{2}} + 1\right)\right)\mathbf{i} + 2\,\mathbf{j} + e^{2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)}\sin 2\left(\tfrac{1}{2}\ln\left(\tfrac{s}{\sqrt{2}} + 1\right)\right)\mathbf{k}$$
$$= \left(\tfrac{s}{\sqrt{2}} + 1\right)\cos\left(\ln\left(\tfrac{s}{\sqrt{2}} + 1\right)\right)\mathbf{i} + 2\,\mathbf{j} + \left(\tfrac{s}{\sqrt{2}} + 1\right)\sin\left(\ln\left(\tfrac{s}{\sqrt{2}} + 1\right)\right)\mathbf{k}$$

19. (a) $\mathbf{r}(t) = \langle\sqrt{2}\,t, e^t, e^{-t}\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle\sqrt{2}, e^t, -e^{-t}\rangle \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$.

Then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{e^t + e^{-t}}\langle\sqrt{2}, e^t, -e^{-t}\rangle = \frac{1}{e^{2t} + 1}\langle\sqrt{2}\,e^t, e^{2t}, -1\rangle \quad \left[\text{after multiplying by } \frac{e^t}{e^t}\right] \quad \text{and}$$

$$\mathbf{T}'(t) = \frac{1}{e^{2t} + 1}\langle\sqrt{2}\,e^t, 2e^{2t}, 0\rangle - \frac{2e^{2t}}{(e^{2t} + 1)^2}\langle\sqrt{2}\,e^t, e^{2t}, -1\rangle$$
$$= \frac{1}{(e^{2t} + 1)^2}\left[(e^{2t} + 1)\langle\sqrt{2}\,e^t, 2e^{2t}, 0\rangle - 2e^{2t}\langle\sqrt{2}\,e^t, e^{2t}, -1\rangle\right] = \frac{1}{(e^{2t} + 1)^2}\langle\sqrt{2}\,e^t\left(1 - e^{2t}\right), 2e^{2t}, 2e^{2t}\rangle$$

Then

$$|\mathbf{T}'(t)| = \frac{1}{(e^{2t} + 1)^2}\sqrt{2e^{2t}(1 - 2e^{2t} + e^{4t}) + 4e^{4t} + 4e^{4t}} = \frac{1}{(e^{2t} + 1)^2}\sqrt{2e^{2t}(1 + 2e^{2t} + e^{4t})}$$
$$= \frac{1}{(e^{2t} + 1)^2}\sqrt{2e^{2t}\left(1 + e^{2t}\right)^2} = \frac{\sqrt{2}\,e^t(1 + e^{2t})}{(e^{2t} + 1)^2} = \frac{\sqrt{2}\,e^t}{e^{2t} + 1}$$

21. $\mathbf{r}(t) = t^3\,\mathbf{j} + t^2\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 3t^2\,\mathbf{j} + 2t\,\mathbf{k}, \quad \mathbf{r}''(t) = 6t\,\mathbf{j} + 2\,\mathbf{k}, \quad |\mathbf{r}'(t)| = \sqrt{0^2 + (3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 4t^2},$

$\mathbf{r}'(t) \times \mathbf{r}''(t) = -6t^2\,\mathbf{i}, \quad |\mathbf{r}'(t) \times \mathbf{r}''(t)| = 6t^2$. Then $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{6t^2}{\left(\sqrt{9t^4 + 4t^2}\right)^3} = \dfrac{6t^2}{(9t^4 + 4t^2)^{3/2}}$.

25. $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. The point $(1, 1, 1)$ corresponds to $t = 1$, and $\mathbf{r}'(1) = \langle 1, 2, 3 \rangle \quad\Rightarrow$

$|\mathbf{r}'(1)| = \sqrt{1 + 4 + 9} = \sqrt{14}$. $\mathbf{r}''(t) = \langle 0, 2, 6t \rangle \quad\Rightarrow\quad \mathbf{r}''(1) = \langle 0, 2, 6 \rangle$. $\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 6, -6, 2 \rangle$, so

$|\mathbf{r}'(1) \times \mathbf{r}''(1)| = \sqrt{36 + 36 + 4} = \sqrt{76}$. Then $\kappa(1) = \dfrac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \dfrac{\sqrt{76}}{\sqrt{14}^3} = \dfrac{1}{7}\sqrt{\dfrac{19}{14}}$.

29. $f(x) = xe^x, \quad f'(x) = xe^x + e^x, \quad f''(x) = xe^x + 2e^x,$

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{|xe^x + 2e^x|}{[1 + (xe^x + e^x)^2]^{3/2}} = \frac{|x + 2|\ e^x}{[1 + (xe^x + e^x)^2]^{3/2}}$$

30. $y' = \dfrac{1}{x}, \quad y'' = -\dfrac{1}{x^2},$

$$\kappa(x) = \frac{|y''(x)|}{\left[1 + (y'(x))^2\right]^{3/2}} = \left|\frac{-1}{x^2}\right| \frac{1}{(1 + 1/x^2)^{3/2}} = \frac{1}{x^2}\,\frac{(x^2)^{3/2}}{(x^2 + 1)^{3/2}} = \frac{|x|}{(x^2 + 1)^{3/2}} = \frac{x}{(x^2 + 1)^{3/2}} \quad [\text{since } x > 0].$$

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$$\kappa'(x) = \frac{(x^2 + 1)^{3/2} - x\left(\frac{3}{2}\right)(x^2 + 1)^{1/2}(2x)}{\left[(x^2 + 1)^{3/2}\right]^2} = \frac{(x^2 + 1)^{1/2}[(x^2 + 1) - 3x^2]}{(x^2 + 1)^3} = \frac{1 - 2x^2}{(x^2 + 1)^{5/2}};$$

$\kappa'(x) = 0 \quad\Rightarrow\quad 1 - 2x^2 = 0$, so the only critical number in the domain is $x = \frac{1}{\sqrt{2}}$. Since $\kappa'(x) > 0$ for $0 < x < \frac{1}{\sqrt{2}}$

and $\kappa'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$, $\kappa(x)$ attains its maximum at $x = \frac{1}{\sqrt{2}}$. Thus, the maximum curvature occurs at $\left(\frac{1}{\sqrt{2}}, \ln\frac{1}{\sqrt{2}}\right)$.

Since $\displaystyle\lim_{x \to \infty} \frac{x}{(x^2 + 1)^{3/2}} = 0$, $\kappa(x)$ approaches 0 as $x \to \infty$.