Section 12.4 - 5, 7, 9, 13, 20, 22, 30, 33, 39, 41

5. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & \frac{1}{2} \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ \frac{1}{2} & 1 \end{vmatrix}\mathbf{k}$

$= \left[-\frac{1}{2} - (-1)\right]\mathbf{i} - \left[\frac{1}{2} - (-\frac{1}{2})\right]\mathbf{j} + \left[1 - (-\frac{1}{2})\right]\mathbf{k} = \frac{1}{2}\mathbf{i} - \mathbf{j} + \frac{3}{2}\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \left(\frac{1}{2}\mathbf{i} - \mathbf{j} + \frac{3}{2}\mathbf{k}\right)\cdot(\mathbf{i} - \mathbf{j} - \mathbf{k}) = \frac{1}{2} + 1 - \frac{3}{2} = 0$ and

$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \left(\frac{1}{2}\mathbf{i} - \mathbf{j} + \frac{3}{2}\mathbf{k}\right)\cdot\left(\frac{1}{2}\mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}\right) = \frac{1}{4} - 1 + \frac{3}{4} = 0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

7. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 1 & 1/t \\ t^2 & t^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1/t \\ t^2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} t & 1/t \\ t^2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} t & 1 \\ t^2 & t^2 \end{vmatrix}\mathbf{k}$

$= (1-t)\,\mathbf{i} - (t-t)\,\mathbf{j} + (t^3 - t^2)\,\mathbf{k} = (1-t)\,\mathbf{i} + (t^3 - t^2)\,\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \langle 1-t, 0, t^3-t^2\rangle\cdot\langle t, 1, 1/t\rangle = t - t^2 + 0 + t^2 - t = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \langle 1-t, 0, t^3-t^2\rangle\cdot\langle t^2, t^2, 1\rangle = t^2 - t^3 + 0 + t^3 - t^2 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

9. According to the discussion preceding Theorem 11, $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, so $(\mathbf{i}\times\mathbf{j})\times\mathbf{k} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$ [by Example 2].

13. (a) Since $\mathbf{b}\times\mathbf{c}$ is a vector, the dot product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is meaningful and is a scalar.

(b) $\mathbf{b}\cdot\mathbf{c}$ is a scalar, so $\mathbf{a}\times(\mathbf{b}\cdot\mathbf{c})$ is meaningless, as the cross product is defined only for two *vectors*.

(c) Since $\mathbf{b}\times\mathbf{c}$ is a vector, the cross product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is meaningful and results in another vector.

(d) $\mathbf{b}\cdot\mathbf{c}$ is a scalar, so the dot product $\mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c})$ is meaningless, as the dot product is defined only for two vectors.

(e) Since $(\mathbf{a}\cdot\mathbf{b})$ and $(\mathbf{c}\cdot\mathbf{d})$ are both scalars, the cross product $(\mathbf{a}\cdot\mathbf{b})\times(\mathbf{c}\cdot\mathbf{d})$ is meaningless.

(f) $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$ are both vectors, so the dot product $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$ is meaningful and is a scalar.

20. By Theorem 8, the cross product of two vectors is orthogonal to both vectors. So we calculate

$$(\mathbf{j}-\mathbf{k})\times(\mathbf{i}+\mathbf{j}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{k} = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

Thus two unit vectors orthogonal to both given vectors are $\pm\frac{1}{\sqrt{3}}(\mathbf{i}-\mathbf{j}-\mathbf{k})$, that is, $\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$ and $-\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$.

22. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle \cdot \langle b_1, b_2, b_3 \rangle = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} b_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} b_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} b_3$$

$$= (a_2 b_3 b_1 - a_3 b_2 b_1) - (a_1 b_3 b_2 - a_3 b_1 b_2) + (a_1 b_2 b_3 - a_2 b_1 b_3) = 0$$

30. (a) $\overrightarrow{PQ} = \langle 4, 2, 3 \rangle$ and $\overrightarrow{PR} = \langle 3, 3, 4 \rangle$, so a vector orthogonal to the plane through P, Q, and R is

$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (2)(4) - (3)(3), (3)(3) - (4)(4), (4)(3) - (2)(3) \rangle = \langle -1, -7, 6 \rangle$ (or any nonzero scalar mutiple thereof).

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = |\langle -1, -7, 6 \rangle| = \sqrt{1 + 49 + 36} = \sqrt{86}$, so the area of triangle PQR is $\frac{1}{2}\sqrt{86}$.

33. By Equation 14, the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product,

which is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = 1(4-2) - 2(-4-4) + 3(-1-2) = 9.$

Thus the volume of the parallelepiped is 9 cubic units.

39. The magnitude of the torque is $|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = (0.18 \text{ m})(60 \text{ N}) \sin(70 + 10)^\circ = 10.8 \sin 80^\circ \approx 10.6 \text{ N·m}$.

41. Using the notation of the text, $\mathbf{r} = \langle 0, 0.3, 0 \rangle$ and $\mathbf{F}$ has direction $\langle 0, 3, -4 \rangle$. The angle θ between them can be determined by

$\cos\theta = \dfrac{\langle 0, 0.3, 0 \rangle \cdot \langle 0, 3, -4 \rangle}{|\langle 0, 0.3, 0 \rangle|\,|\langle 0, 3, -4 \rangle|} \Rightarrow \cos\theta = \dfrac{0.9}{(0.3)(5)} \Rightarrow \cos\theta = 0.6 \Rightarrow \theta \approx 53.1^\circ$. Then $|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta \Rightarrow$

$100 = 0.3\,|\mathbf{F}| \sin 53.1^\circ \Rightarrow |\mathbf{F}| \approx 417 \text{ N}$.