Section 12.3 - 1, 7, 9, 20, 23(a,b), 27, 31, 42, 47, 49, 50

1. (a) $\mathbf{a} \cdot \mathbf{b}$ is a scalar, and the dot product is defined only for vectors, so $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ has no meaning.

   (b) $(\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$ is a scalar multiple of a vector, so it does have meaning.

   (c) Both $|\mathbf{a}|$ and $\mathbf{b} \cdot \mathbf{c}$ are scalars, so $|\mathbf{a}|\,(\mathbf{b} \cdot \mathbf{c})$ is an ordinary product of real numbers, and has meaning.

7. $\mathbf{a} \cdot \mathbf{b} = (2\,\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} - \mathbf{j} + \mathbf{k}) = (2)(1) + (1)(-1) + (0)(1) = 1$

9. By Theorem 3, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta = (6)(5)\cos\frac{2\pi}{3} = 30\left(-\frac{1}{2}\right) = -15.$

20. $|\mathbf{a}| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$, $|\mathbf{b}| = \sqrt{4^2 + 0^2 + (-3)^2} = \sqrt{25} = 5$, and

    $\mathbf{a} \cdot \mathbf{b} = (1)(4) + (2)(0) + (-2)(-3) = 10$. Then $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{10}{3 \cdot 5} = \dfrac{2}{3}$ and $\theta = \cos^{-1}\left(\frac{2}{3}\right) \approx 48°$.

23. (a) $\mathbf{a} \cdot \mathbf{b} = (-5)(6) + (3)(-8) + (7)(2) = -40 \neq 0$, so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Also, since $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

    (b) $\mathbf{a} \cdot \mathbf{b} = (4)(-3) + (6)(2) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

27. Let $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ be a vector orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$. Then $\mathbf{a} \cdot (\mathbf{i} + \mathbf{j}) = 0 \quad \Leftrightarrow \quad a_1 + a_2 = 0$ and $\mathbf{a} \cdot (\mathbf{i} + \mathbf{k}) = 0 \quad \Leftrightarrow \quad a_1 + a_3 = 0$, so $a_1 = -a_2 = -a_3$. Furthermore $\mathbf{a}$ is to be a unit vector, so $1 = a_1^2 + a_2^2 + a_3^2 = 3a_1^2$ implies $a_1 = \pm\frac{1}{\sqrt{3}}$. Thus $\mathbf{a} = \frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}$ and $\mathbf{a} = -\frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}$ are two such unit vectors.

31. The curves $y = x^2$ and $y = x^3$ meet when $x^2 = x^3 \quad \Leftrightarrow \quad x^3 - x^2 = 0 \quad \Leftrightarrow \quad x^2(x-1) = 0 \quad \Leftrightarrow \quad x = 0, x = 1$. We have $\dfrac{d}{dx}x^2 = 2x$ and $\dfrac{d}{dx}x^3 = 3x^2$, so the tangent lines of both curves have slope 0 at $x = 0$. Thus the angle between the curves is $0°$ at the point $(0,0)$. For $x = 1$, $\dfrac{d}{dx}\,x^2\Big|_{x=1} = 2$ and $\dfrac{d}{dx}\,x^3\Big|_{x=1} = 3$ so the tangent lines at the point $(1,1)$ have slopes 2 and 3. Vectors parallel to the tangent lines are $\langle 1, 2\rangle$ and $\langle 1, 3\rangle$, and the angle θ between them is given by

$$\cos\theta = \frac{\langle 1,2\rangle \cdot \langle 1,3\rangle}{|\langle 1,2\rangle|\,|\langle 1,3\rangle|} = \frac{1+6}{\sqrt{5}\,\sqrt{10}} = \frac{7}{5\sqrt{2}}$$

    Thus $\theta = \cos^{-1}\left(\dfrac{7}{5\sqrt{2}}\right) \approx 8.1°$.

42. $|\mathbf{a}| = \sqrt{4 + 9 + 36} = 7$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{1}{7}(-10 - 3 - 24) = -\frac{37}{7}$, while the vector projection is $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = -\dfrac{37}{7}\dfrac{\mathbf{a}}{|\mathbf{a}|} = -\frac{37}{7} \cdot \frac{1}{7}\langle -2, 3, -6\rangle = -\frac{37}{49}\langle -2, 3, -6\rangle = \left\langle \frac{74}{49}, -\frac{111}{49}, \frac{222}{49}\right\rangle$.

47. $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = 2 \quad\Leftrightarrow\quad \mathbf{a}\cdot\mathbf{b} = 2\,|\mathbf{a}| = 2\sqrt{10}$. If $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then we need $3b_1 + 0b_2 - 1b_3 = 2\sqrt{10}$.

One possible solution is obtained by taking $b_1 = 0$, $b_2 = 0$, $b_3 = -2\sqrt{10}$. In general, $\mathbf{b} = \left\langle s, t, 3s - 2\sqrt{10}\right\rangle$, $s, t \in \mathbb{R}$.

49. The displacement vector is $\mathbf{D} = (6-0)\,\mathbf{i} + (12-10)\,\mathbf{j} + (20-8)\,\mathbf{k} = 6\,\mathbf{i} + 2\,\mathbf{j} + 12\,\mathbf{k}$ so, by Equation 12, the work done is $W = \mathbf{F}\cdot\mathbf{D} = (8\,\mathbf{i} - 6\,\mathbf{j} + 9\,\mathbf{k})\cdot(6\,\mathbf{i} + 2\,\mathbf{j} + 12\,\mathbf{k}) = 48 - 12 + 108 = 144$ joules.

50. Here $|\mathbf{D}| = 1000$ m, $|\mathbf{F}| = 1500$ N, and $\theta = 30^\circ$. Thus

$W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}|\,|\mathbf{D}|\cos\theta = (1500)(1000)\left(\frac{\sqrt{3}}{2}\right) = 750{,}000\,\sqrt{3}$ joules.