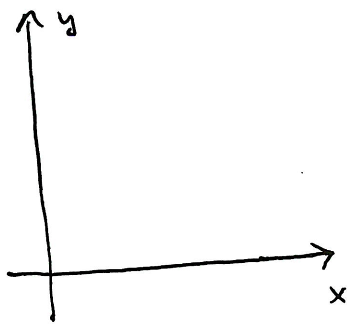
Section 16.4 - Green's Theorem

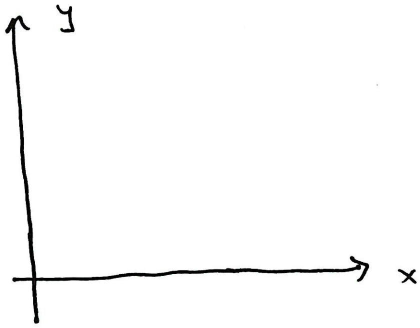
For Conservative vector fields have FTC for Line integrals

★ Now want something for non-Conservative vector fields

- Positive Orientation:

  For simple closed curves <u>positive orientation</u> refers to





- Notation: C a simple closed curve

$$\int_C \vec{F}\cdot d\vec{r}$$

$$\int_{-C} \vec{F}\cdot d\vec{r}$$

∂D means boundary of D

Green's Theorem Let C be a positively oriented, piecewise smooth, Simple closed curve. C bounds D i.e. $\partial D = C$
P, Q have continuous first order partials on D

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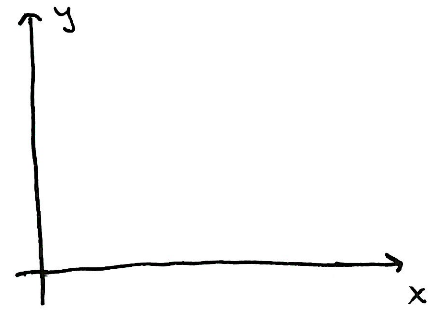
- **Proof:** Any Curve C piecewise smooth, Simple closed Can be broken into rectangles or as follows:



Show: ① $\int_C P\,dx = \iint_D -\frac{\partial P}{\partial y}\,dA$

② $\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$

$\boxed{\text{Example}}$ Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve from (0,0) to (1,0) to (0,1) using (a) Green's theorem and (b) Line Integrals.

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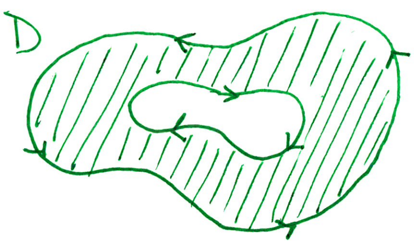
- Reverse Application of Green's theorem:

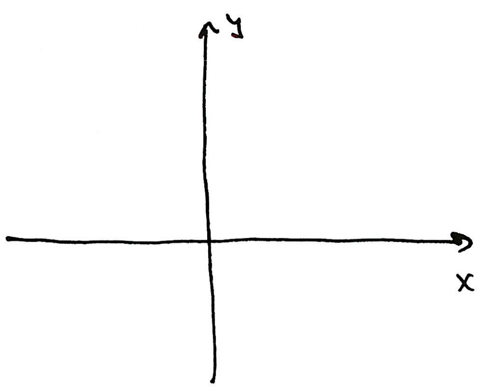
$$A = \text{Area of } D = \iint_D 1\, dA \quad \text{write as line integrals.}$$

$\boxed{\text{Example}}$ Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

- Green's Theorem on Regions with holes:



$\boxed{\text{Example}}$ $\vec{F}(x,y) = \frac{\langle -y, x\rangle}{|\langle -y, x\rangle|^2}$ show $\int_C \vec{F} \cdot d\vec{r} = 2\pi$ for every positively oriented simple closed path around the origin.



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Recall: $\boxed{\text{Theorem (16.3)}}$ $\vec{F} = \langle P, Q \rangle$ on open simply-connected region D.
P, Q have continuous first order partial Derivatives
with $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ on D then $\vec{F}$ is Conservative.

- Proof:

C simple closed curve in D with region R bounded by C

By Green's Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} =$$

★ Any closed curve can be broken into simple closed curves.

So $\int_C \vec{F} \cdot d\vec{r} =$

$\Rightarrow$ $\int_C \vec{F} \cdot d\vec{r}$ is

$\Rightarrow$ By FTC for line Integrals $\vec{F}$ is