Section 13.3 – Arc Length & Curvature

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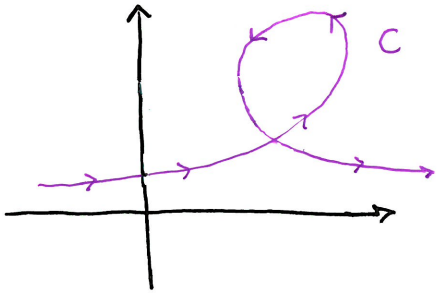
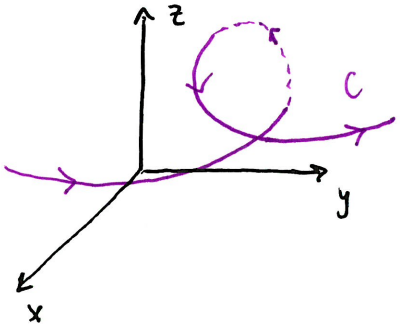
- Arc length of a curve C with parametric equations $x = f(t)$ and $y = g(t)$ (Section 10.2)





Example 1 Find the length of $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ from $(1,0,0)$ to $(1, 0, 2\pi)$

- Parametrize a curve with respect to Arc length:

Why? 1)

2)

Idea: $S(t)$ is arc length as a function of t

$t(s) = S^{-1}$ is time as a function of S

Example 2 Reparametrize the helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ wrt to arc length measured from $(1,0,0)$ in the direction of increasing t.

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- smooth parametrization $\vec{r}(t)$:
- smooth curve C:
- Curvature:

★ Curvature of the Earth: thiscolossal.com/wp-content/uploads/2018/01/roads2.gif

Example 3 Show that the curvature of a circle of radius a is $1/a$.

Theorem $$K(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

Proof: $\vec{T} = \frac{\vec{r}'}{|\vec{r}'|}$ $|\vec{r}'| = \frac{ds}{dt}$ so $\vec{r}' = \frac{ds}{dt}\vec{T}$

Differentiate wrt t using product rule

$\vec{r}'' = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}'$ $\vec{r}' \times \vec{r}'' = \vec{r}' \times \frac{d^2s}{dt^2}\vec{T} + \vec{r}' \times \frac{ds}{dt}\vec{T}'$ ($\vec{r}' \times \vec{T} = \vec{0}$)

$|\vec{r}' \times \vec{r}''| = |\vec{r}' \times |\vec{r}'|\vec{T}'| = ||\vec{r}'|^2 \vec{T} \times \vec{T}'|$

$= |\vec{r}'|^2 |\vec{T}||\vec{T}'| \sin\pi/2$ $[\vec{T} \perp \vec{T}' \Rightarrow \theta = \pi/2]$

Dividing both sides by $|\vec{r}'|^3$ gives:

$$\frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{|\vec{T}'|}{|\vec{r}'|} = K(t)$$ ∎

Example 4 Find the curvature of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $(0,0,0)$.

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- Curvature of a function $y = f(x)$:

- Normal & Binormal vectors: Note $\vec{T}' \perp \vec{T}$

  Unit Normal vector: $\vec{N}(t) =$

  Binormal vector: $\vec{B}(t) =$

  Normal plane to C at a point P:

Example 7 Find the equation of the normal plane to $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ at $(0, 1, \pi/2)$

- Question: Why not talk about a tangent plane to C at point p?

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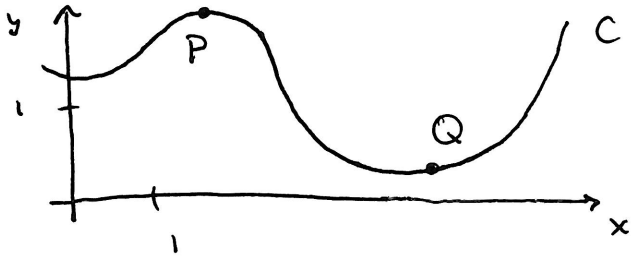
- Extra Examples

#31 At what point does $y = e^x$ have max curvature? What happens to the curvature as $x \to \infty$?

#33 (a) Is the curvature at P or Q greater? Explain



#46 Consider the curvature of the family of functions $y = e^{cx}$ at $x = 0$. For which members is $K(0)$ largest?

#53 At what point on the curve $x = t^3$, $y = 3t$, $z = t^4$ is the normal plane parallel to the plane $6x + 6y - 8z = 1$?