Section 12.4 - The Cross Product

Recall: dot product of two vectors produced a scalar
Would like a product that is meaningful and produces a vector

- Cross Product: $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{a} \times \vec{b} =$

Better way to Compute: use the Determinant of a 3×3 matrix

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

Example 1 Compute $\vec{a} \times \vec{b}$ for $\vec{a} = \langle 1, 3, 4 \rangle$, $\vec{b} = \langle 2, 7, -5 \rangle$

Theorem The vector $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Theorem If θ is the angle between $\vec{a}$ and $\vec{b}$ then $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$.

Corollary Two nonzero vectors $\vec{a}, \vec{b}$ are parallel iff $\vec{a} \times \vec{b} = \vec{0}$.

Questions: ① What does $|\vec{a} \times \vec{b}|$ represent geometrically in relation to $\vec{a}$ and $\vec{b}$?
② Do you think the cross product is commutative: $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$?

Example 4 Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.

Warning:

- Compute using the Right Hand Rule:

$\vec{i} \times \vec{j} =$ $\qquad$ $\vec{j} \times \vec{k} =$ $\qquad$ $\vec{i} \times \vec{k} =$

$\vec{j} \times \vec{i} =$ $\qquad$ $\vec{k} \times \vec{j} =$ $\qquad$ $\vec{k} \times \vec{i} =$

Theorem

1. $\vec{a} \times \vec{b} =$
2. $(c\vec{a}) \times \vec{b} =$
3. $\vec{a} \times (\vec{b} + \vec{c}) =$
4. $(\vec{a} + \vec{b}) \times \vec{c} =$
5. ★ $\vec{a} \cdot (\vec{b} \times \vec{c}) =$
6. $\vec{a} \times (\vec{b} \times \vec{c}) =$

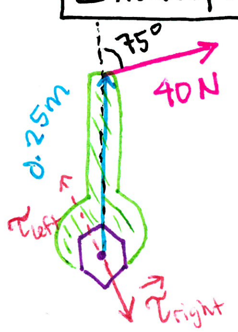
★ Scalar Triple Product: $\vec{a} \cdot (\vec{b} \times \vec{c}) =$

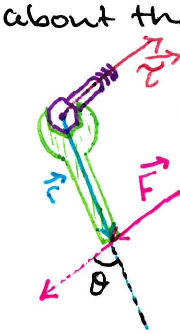
Geometrically:

Example 5 Use the scalar triple product to show $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$, $\vec{c} = \langle 0, -9, 18 \rangle$ are coplanar (all in the same plane).

- Application: Torque ($\vec{\tau}$) - measuring the tendency of the body to rotate about the origin, when a force ($\vec{F}$) acts on the rigid body. (position $\vec{r}$)

Example 6 A bolt is tightened by applying 40N of force to a 0.25m wrench. Find the magnitude of the torque about the bolt center.





$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

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- Extra Examples:

#42. Let $\vec{v} = 5\vec{j}$ and let $\vec{u}$ be a vector with length 3 that starts at the origin and rotates in the xy plane. Find the max and min values of $|\vec{u} \times \vec{v}|$. In what direction does $\vec{u} \times \vec{v}$ point?

#45(a). Let P be a point not on the line L, passing through points Q and R. Show that the distance d from P to L is $d = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|}$ where $\vec{a} = \vec{QR}$ and $\vec{b} = \vec{QP}$.

#48. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ show that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

#53. Suppose that $\vec{a} \neq \vec{0}$.

(a) If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ does it follow $\vec{b} = \vec{c}$?

(b) If $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ does it follow $\vec{b} = \vec{c}$?

(c) If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ does it follow $\vec{b} = \vec{c}$?