Section 16.1 - 16.3 Review

- **Section 16.1 — Vector fields**

A vector field is a function $\vec{F}$ that assigns to each point a vector (in $\mathbb{R}^n$)

Ex. $\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k} = \langle P, Q, R \rangle$

★ Gradient vector field

$$\nabla f(x,y,z) = f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k} = \langle f_x, f_y, f_z \rangle$$

Definitions - a vector field $\vec{F}$ is Conservative if it is the gradient of a scalar function. That is if there exists f so that $\vec{F} = \nabla f$; if so then f is called a potential function for $\vec{F}$.

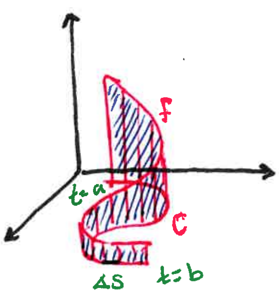
Ex. Find a potential function for the conservative vector field $\vec{F} = \langle \sin x, 2\cos y \rangle$

$f_x = \sin x \qquad f = \int \sin x \, dx = -\cos x + g(y)$

$f_y = 2\cos y \qquad f = \int 2\cos y \, dy = 2\sin y + h(x)$

$$\boxed{f = -\cos x + 2\sin y + C}$$

- **Section 16.2 — Line Integrals**



Compute area of ribbon - with respect to the arc length

Smooth C: $x = x(t) \quad y = y(t) \quad a \le t \le b$

$\Delta S = \sqrt{(x'(t))^2 + y'(t)^2}\,\Delta t$ - arc length

Area of a rectangle $= f(x,y) \cdot \Delta S$

Line integral of f along C $= \boxed{\int_C f \, ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$

★ Independent of direction $\int_C f\,ds = \int_{-C} f\,ds$

Line integrals with respect to x, y:

★ Direction matters sign changes

$$\boxed{\begin{aligned} \int_C f(x,y)\,dx &= \int_a^b f(x(t),y(t))\,\frac{dx}{dt}\cdot dt \\ \int_C f(x,y)\,dy &= \int_a^b f(x(t),y(t))\,\frac{dy}{dt}\cdot dt \end{aligned}}$$

Notation:

$\int_C P(x,y)\,dx + Q(x,y)\,dy$

Line integrals in space: $\int_C f(x,y,z)\,ds = \int_a^b f(r(t))|\vec{r}'(t)|\,dt$ where $C: \vec{r}(t) \quad a \le t \le b$

Line integrals of vector fields - Example Work along a curve C in $\mathbb{R}^3$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(r(t))\cdot \vec{r}'(t)\,dt = \int_C \vec{F}\cdot\vec{T}\,ds = \int_C P\,dx + Q\,dy + R\,dz}$$

where $\vec{F} = \langle P, Q, R \rangle$

Ex. Find the work done by the force $\vec{F} = \langle x^2, ye^x \rangle$ on a particle that moves along $x = y^2 + 1$ from $(1,0)$ to $(2,1)$.

$$W = \int_C \vec{F} \cdot d\vec{r} \qquad C: \; x = t^2 \quad y = t \qquad 0 \le t \le 1$$

$$= \int_0^1 \langle t^4, te^{t^2} \rangle \cdot \langle 2t, 1 \rangle \, dt$$

$$= \int_0^1 2t^5 + te^{t^2} \, dt = \frac{1}{3}t^6 + \frac{1}{2}e^{t^2} \Big|_0^1 = \boxed{\frac{1}{3} + \frac{e}{2}}$$

- **Section 16.3 - The Fundamental Theorem for Line integrals**

C - Smooth Curve given by $\vec{r}(t)$, $a \le t \le b$

f differentiable with ∇f continuous on C then $\boxed{\int_C \nabla f \cdot d\vec{r} = f(r(b)) - f(r(a))}$

Path independence: $\int_C \vec{F} \cdot d\vec{r}$ is <u>independent of path</u> if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths in D with same start and end.

Theorems:

4. $\int_C \vec{F} \cdot d\vec{r}$ independent of path $\Rightarrow$ $\vec{F}$ Conservative
5. $\vec{F} = \langle P, Q \rangle$ conservative P, Q have continuous partials $\Rightarrow$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
6. $\vec{F} = \langle P, Q \rangle$ on open simply connected D, P, Q continuous partials and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\Rightarrow$ $\vec{F}$ Conservative

Ex. Show $\vec{F}$ is conservative, find f so that $\nabla f = \vec{F}$ and compute $\int_C \vec{F} \cdot d\vec{r}$.

$$\vec{F} = \langle xy^2, \; x^2y + 1 \rangle \qquad C: \vec{r}(t) = \langle \cos t, 2\sin t \rangle \quad 0 \le t \le \pi/2$$

$\frac{\partial P}{\partial y} = 2xy$

$\frac{\partial Q}{\partial x} = 2xy$ ✓

both continuous

$\vec{F}$ is Conservative

$$f = \int xy^2 \, dx = \frac{x^2y^2}{2} + g(y)$$

$$f = \int x^2y + 1 \, dy = \frac{x^2y^2}{2} + y + h(x)$$

$$f = \frac{x^2y^2}{2} + y$$

by FTC

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(r(\pi/2)) - f(r(0))$$

$$= f(0, 2) - f(1, 0) = 2 - 0 = \boxed{2}$$