Section 16.9 - The Divergence Theorem

Stoke's Theorem allows us to write a Line Integral of a vector Field as a Double Integral of a Scalar Function.

Want to be able to write a Surface Integral of a vector Field as a Triple Integral? of a Scalar Function?.

$d\vec{S} = \vec{n}\,dS$ Extension of $dS = \Delta$ surface area

$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds$ where $\vec{F}\cdot\vec{T}$ is the tangential component of $\vec{F}$

Now would like a line integral of the normal component of $\vec{F}$: $\vec{F}\cdot\vec{n}$

Note: $\vec{T}\,ds = \langle dx, dy\rangle$ so $\vec{n}\,ds = \langle dy, -dx\rangle$

$$\boxed{\int_C \vec{F}\cdot\vec{n}\,ds} = \int_C \langle P, Q\rangle\cdot\langle dy, -dx\rangle = \int_C P\,dy - Q\,dx$$

Stoke's/Green's Theorem

$$= \iint_D \frac{\partial P}{\partial x} - \left(-\frac{\partial Q}{\partial y}\right) dA = \boxed{\iint_D \operatorname{div}\vec{F}\,dA}$$

The Divergence Theorem:

- E simple solid region
- $S = \partial E$ with positive orientation
- $\vec{F}$ components with continuous partials on open region containing E

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV$$

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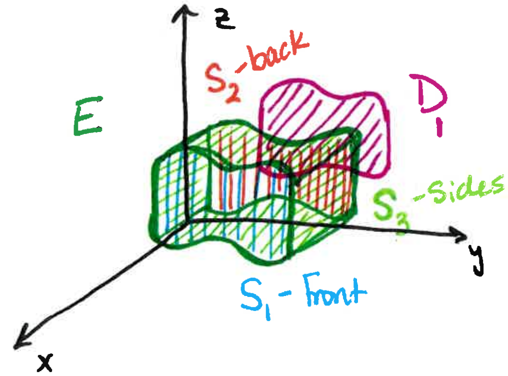


Proof Overview: $\vec{F} = \langle P, Q, R \rangle$

Assume:

$$E = \{(x,y,z) \mid (y,z) \in D_1,\ g_1(y,z) \le x \le g_2(y,z)\}$$
$$= \{(x,y,z) \mid (x,z) \in D_2,\ h_1(x,z) \le y \le h_2(x,z)\}$$
$$= \{(x,y,z) \mid (x,y) \in D_3,\ k_1(x,y) \le z \le k_2(x,y)\}$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_S \langle P, Q, R \rangle \cdot \vec{n}\, dS$$
$$= \iint_S P\vec{i} \cdot \vec{n}\, dS + \iint_S Q\vec{j} \cdot \vec{n}\, dS + \iint_S R\vec{k} \cdot \vec{n}\, dS$$

$$\iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E \frac{\partial P}{\partial x}\, dV + \iiint_E \frac{\partial Q}{\partial y}\, dV + \iiint_E \frac{\partial R}{\partial z}\, dV$$

Enough to Show: $\displaystyle \iint_S P\vec{i} \cdot \vec{n}\, dS = \iiint_E \frac{\partial P}{\partial x}\, dV$

$$\iiint_E \frac{\partial P}{\partial x}\, dV = \iint_{D_1} \left[\int_{g_1}^{g_2} \frac{\partial P}{\partial x}\, dx \right] dA = \iint_{D_1} \left[P(g_2, y, z) - P(g_1, y, z) \right] dA$$

$\vec{n}_1 \parallel \langle 1, -g_{2y}, -g_{2z} \rangle$

$\vec{n}_2 \parallel \langle -1, +g_{1y}, +g_{1z} \rangle$

$\vec{n}_3 = \langle 0, a, b \rangle$

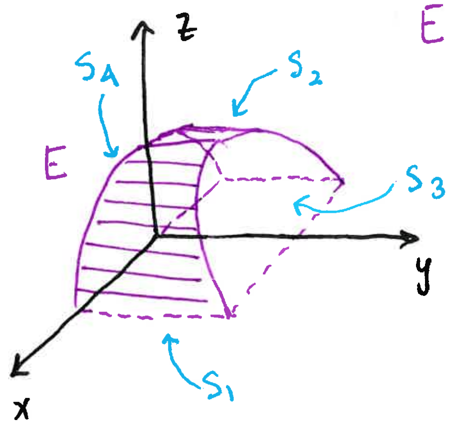
$$\iint_S P\vec{i} \cdot \vec{n}\, dS = \iint_{S_1} P\vec{i} \cdot \vec{n}_1\, dS + \iint_{S_2} P\vec{i} \cdot \vec{n}_2\, dS + \cancelto{0}{\iint_{S_3} P\vec{i} \cdot \vec{n}_3\, dS}$$
$$= \iint_{D_1} P(g_2, y, z)\, dA - \iint_{D_1} P(g_1, y, z)\, dA \quad \blacksquare$$

Example Find the flux of the vector field $\vec{F} = \langle x, y, z \rangle$ over the unit sphere.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E \left(\frac{d}{dx}(x) + \frac{d}{dy}(y) + \frac{d}{dz}(z) \right) dV = \iiint_E 3\, dV$$
$$= 3 \cdot \frac{4}{3} \pi (1)^3 = \boxed{4\pi}$$

Section 16.9 – The Divergence Theorem

Example Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle xy,\ (y^2 + e^{xz^2}),\ \sin(xy)\rangle$ and S is the surface of E bounded by $z = 1 - x^2$, $z = 0$, $y = 0$ and $y + z = 2$.



$E = \{(x,y,z) \mid 0 \le y \le 2 - z,\ 0 \le z \le 1 - x^2,\ -1 \le x \le 1\}$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV$$

$$= \int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z}\left(\frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + e^{xz^2}\right) + \frac{\partial}{\partial z}(\sin(xy))\right)dy\,dz\,dx$$

$$= \int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z}(y + 2y)\,dy\,dz\,dx$$

$$= \int_{-1}^{1}\int_0^{1-x^2}\frac{3}{2}(2-z)^2\,dz\,dx$$

$$= \int_{-1}^{1} -\frac{1}{2}(2-z)^3\Big|_0^{1-x^2}dx = -\frac{1}{2}\int_{-1}^{1}(1+x^2)^3 - 8\,dx$$

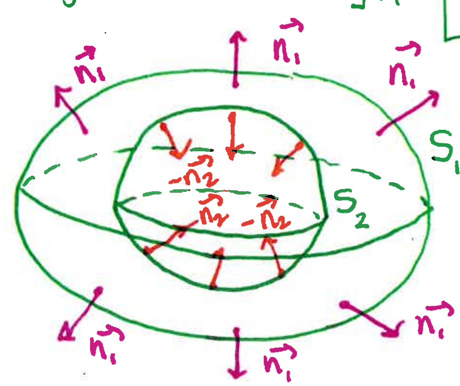
$$= -\frac{1}{2}\left[x + x^3 + \frac{3}{5}x^5 + \frac{1}{7}x^7 - 8x\right]_{-1}^{1} = \boxed{\frac{184}{35}}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1}\vec{F}\cdot d\vec{S} + \iint_{S_2}\vec{F}\cdot d\vec{S} + \iint_{S_3}\vec{F}\cdot d\vec{S} + \iint_{S_4}\vec{F}\cdot d\vec{S}$$

Yikes!

- Hallow Solids: $\partial E = S = S_1 \cup S_2$

Normal to E is $\vec{n} = \begin{cases}\vec{n}_1 & \text{on } S_1\\ -\vec{n}_2 & \text{on } S_2\end{cases}$



$$\iiint_E \operatorname{div}\vec{F}\,dV = \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS$$

$$= \iint_{S_1}\vec{F}\cdot\vec{n}_1\,dS + \iint_{S_2}\vec{F}\cdot(-\vec{n}_2)\,dS$$

$$= \boxed{\iint_{S_1}\vec{F}\cdot d\vec{S} - \iint_{S_2}\vec{F}\cdot d\vec{S}}$$