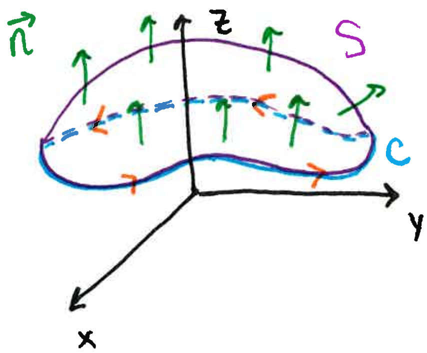
Section 16.8 – Stoke's Theorem

★ Green's Theorem for vector Functions of 3-variables

- Green's theorem relates double Integral over domain $D \subseteq \mathbb{R}^2$
  to Line Integral over boundary $C = \partial D \subseteq \mathbb{R}^2$
- Stoke's Theorem relates Surface Integral over Surface $S \subseteq \mathbb{R}^3$
  to Line Integral over boundary $C = \partial S \subseteq \mathbb{R}^3$

Important: Orientation of S induces orientation on $\partial S = C$



Positive Orientation of S induces Positive orientation on $C = \partial S$

★ C positively oriented, as you walk around C, Surface S is always on your left!

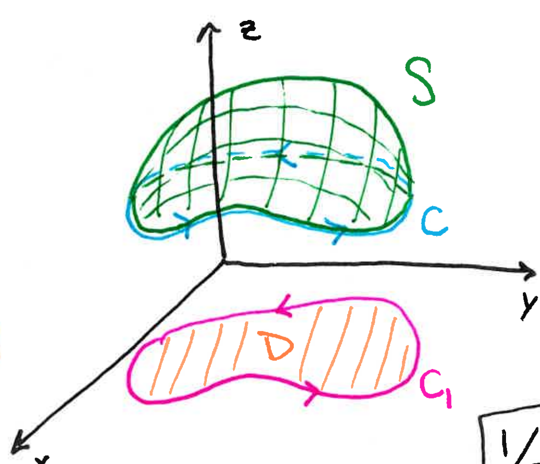
Stoke's Theorem

- S Oriented Piecewise-smooth Surface
- $C = \partial S$ Simple, closed, piecewise-smooth, positive orientation
- $\vec{F}$ vector field, components having continuous partials on open region of $\mathbb{R}^3$ containing S

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

$$\| \qquad\qquad \|$$

$$\underbrace{\int_{C_1} \vec{F}(r(t)) \cdot \vec{r}\,'(t)\, dt}_{\text{Line Integral Def}} = \underbrace{\iint_D \operatorname{curl} \vec{F}(r(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA}_{\text{Surface Integral Def}}$$



Proof (special case): $S: z=g(x,y)$ with $(x,y) \in D \subseteq \mathbb{R}^2$

$C = \partial S \subseteq \mathbb{R}^3$ and $C_1 = \partial D \subseteq \mathbb{R}^2$

$\vec{F} = \langle P, Q, R \rangle$ $\quad$ $\text{Curl}\,\vec{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$

$\vec{n} = \langle -g_x, -g_y, 1 \rangle$

$$\iint_S \text{Curl}\,\vec{F}\cdot d\vec{S} = \iint_D \text{Curl}\,\vec{F}\cdot\vec{n}\,dA = \iint \text{Curl}\,\vec{F}\cdot\langle -g_x, -g_y, 1\rangle\,dA$$

$$= \iint_D \left[\underline{(Q_z - R_y)g_x} + \underline{(R_x - P_z)g_y} + \underline{(Q_x - P_y)} \right] dA$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,\underline{dz} \qquad dz = g_x\,dx + g_y\,dy$$

$$= \int_C (P + Rg_x)\,dx + (Q + Rg_y)\,dy$$

$\boxed{\text{Green's}}$
$$= \iint_D \left(\frac{\partial}{\partial x}(Q + Rg_y) - \frac{\partial}{\partial y}(P + Rg_x) \right) dA$$

★ Remember P, Q, R are functions of $x, y, z = g(x,y)$

$$= \iint_D \left(\underline{\frac{\partial}{\partial x}(Q)} + \frac{\partial}{\partial x}(Rg_y) - \underline{\frac{\partial}{\partial y}(P)} - \frac{\partial}{\partial y}(Rg_x) \right) dA$$

$$= \iint_D \left(Q_x + Q_z g_x + \frac{\partial}{\partial x}(R) g_y + \cancel{R g_{yx}} - P_y - P_z g_y - \frac{\partial}{\partial y}(R) g_x - \cancel{R g_{xy}} \right) dA$$

By Clairaut's Theorem $g_{yx} = g_{xy}$

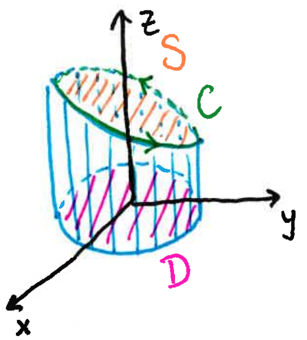
$$= \iint_D \left[\underline{Q_x} + \underline{Q_z g_x} + (\underline{R_x} + \cancel{R_z g_x)g_y} - \underline{P_y} - \underline{P_z g_y} - (\underline{R_y g_x} + \cancel{R_z g_y g_x}) \right] dA$$

$$= \iint_D \left[\underline{(Q_z - R_y)g_x} + \underline{(R_x - P_z)g_y} + \underline{(Q_x - P_y)} \right] dA \quad \blacksquare$$

Section 16.8 - Stoke's Theorem

MVC

Ex. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = \langle -y^2, x, z^2 \rangle$ and C is the curve of intersection of $y+z=2$ and $x^2+y^2=1$; orient C to be counterclockwise when viewed from above.



$S: \; z = 2-y$

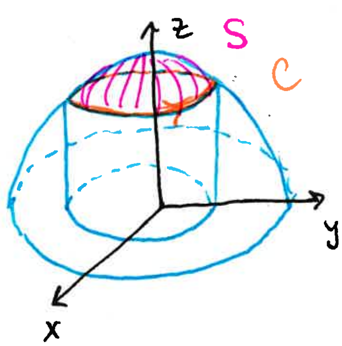
$\text{Curl}\,\vec{F} = \langle 0, 0, 2y+1 \rangle \qquad \vec{n} = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle = \langle 0, 1, 1 \rangle$

$$\int_C \vec{F}\cdot d\vec{r} \overset{\text{Stoke's}}{=} \iint_S \text{Curl}\,\vec{F}\cdot d\vec{S} = \iint_D \langle 0,0,2y+1\rangle \cdot \langle 0,1,1\rangle \, dA$$

$$= \iint_D 2y+1 \, dA \qquad D: \; x = r\cos\theta \quad y = r\sin\theta \quad 0 \le r \le 1 \quad 0 \le \theta \le 2\pi$$

$$= \int_0^{2\pi}\int_0^1 (2r\sin\theta + 1)\, r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{2} + \frac{2}{3}\sin\theta\right)d\theta = \boxed{\pi}$$

Ex. Use Stoke's Theorem to compute the integral $\iint_S \text{Curl}\,\vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle xz, yz, xy \rangle$ and $S: x^2+y^2+z^2 = 4$ and $x^2+y^2 = 1$ above xy-plane



$C: \; 1 + z^2 = 4 \qquad z = \sqrt{3} \qquad x = \cos\theta \quad y = \sin\theta \quad 0 \le \theta \le 2\pi$

$$\iint_S \text{Curl}\,\vec{F}\cdot d\vec{S} \overset{\text{Stoke's}}{=} \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} \vec{F}\cdot \vec{r}\,'(\theta)\, d\theta$$

$$= \int_0^{2\pi} \langle \sqrt{3}\cos\theta, \sqrt{3}\sin\theta, \sin\theta\cos\theta \rangle \cdot \langle -\sin\theta, \cos\theta, 0 \rangle \, d\theta$$

$$= \int_0^{2\pi} \left(-\sqrt{3}\cos\theta\sin\theta + \sqrt{3}\sin\theta\cos\theta\right) d\theta$$

$$= \int_0^{2\pi} 0 \, d\theta = \boxed{0}$$