Section 16.7 - Surface Integrals of Vector Fields

- Applications:
  1. Gravitational & Pressure Forces
  2. Fluid Flow / mass Flow across a surface
  3. Electric charge & Electric Fields

Recap:

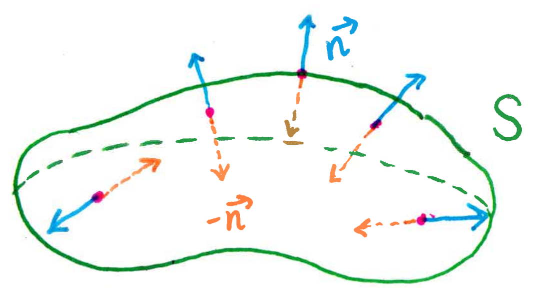
	Scalar Functions	Vector Functions
Line Integrals	$\int_C f\,ds = \int_a^b f(r(t))\,\|\vec{r}\,'(t)\|\,dt$	$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(r(t))\cdot \vec{r}\,'(t)\,dt$
Surface Integrals	$\iint_S f\,dS = \iint_D f(r(u,v))\,\|\vec{r}_u \times \vec{r}_v\|\,dA$	Guess: $\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(r(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\,dA$

• Orientation of Surfaces

★ Make a Möbius strip - Color each side a different color

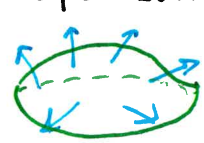
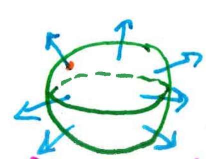
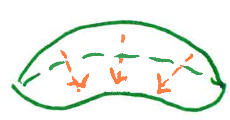
↳ Surface having only one side!

No top/bottom → Non-orientable



- S is <u>orientable</u> if there is a unit normal vector $\vec{n}$ at every point with $\vec{n}$ varying continuously over S.

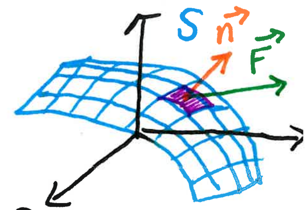
- S has an <u>orientation</u> when $\vec{n}$ or $-\vec{n}$ is chosen.

	Open Surface	Closed surface
Positive Orientation:	 $\vec{k}$ component > 0	 $\vec{n}$ points outwards
Negative Orientation:	 $\vec{k}$ component < 0	 $\vec{n}$ points inward

Section 16.7 - Surface Integrals of Vector Functions

MVC

$\vec{n}$ unit normal vector



- Fluid Flow Motivation:

Fluid with density ρ and velocity field $\vec{V}$ flowing through S

Rate of flow per unit area: $\vec{F} = \rho \vec{V}$

How much of $\vec{F}$ goes in $\vec{n}$ direction

Mass of fluid per unit time crossing S in direction $\vec{n}$: $\underbrace{(\vec{F}\cdot\vec{n})}_{}\Delta S$

Rate of flow through S: $\iint_S \vec{F}\cdot\vec{n}\,dS$ ← Called the Flux of $\vec{F}$ accross S or the Surface Integral of $\vec{F}$

- Surface Integrals of Vector Fields:

$\vec{F}$ continuous, defined on an oriented surface S with unit normal $\vec{n}$

then the Surface integral of $\vec{F}$ over S:

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS$$

- S parametrized by $\vec{r}(u,v)$ then: $\vec{n} = \vec{r}_u \times \vec{r}_v / |\vec{r}_u \times \vec{r}_v|$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot \frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\cdot dS = \iint_D \vec{F}\cdot \frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\cdot |\vec{r}_u\times\vec{r}_v|\,dA_{uv}$$

$$= \boxed{\iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA_{uv}}$$

- S given by $z = g(x,y)$ then: $\vec{n} = \langle -g_x, -g_y, 1\rangle$ Not a unit vector

$\vec{F} = \langle P, Q, R\rangle$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot\vec{n}\,dA = \boxed{\iint_D (-Pg_x - Qg_y - R)\,dA}$$

Example Find the flux of the vector field $\vec{F} = \langle z, y, x \rangle$ across the sphere $S: x^2 + y^2 + z^2 = 1$

$\vec{r}(\varphi, \theta) = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle \quad 0 \le \varphi \le \pi \quad 0 \le \theta \le 2\pi$

$\vec{r}_\varphi \times \vec{r}_\theta = \langle \sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi \rangle$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^\pi \int_0^{2\pi} \vec{F}(\vec{r}(\varphi,\theta)) \cdot (\vec{r}_\varphi \times \vec{r}_\theta)\, d\theta\, d\varphi$$

$$= \int_0^\pi \int_0^{2\pi} \left(\cancel{\cos\varphi\sin^2\varphi\cos\theta}^{\,0} + \sin^3\varphi\sin^2\theta + \cancel{\sin^2\varphi\cos\varphi\cos\theta}^{\,0}\right) d\theta\, d\varphi$$

$$= \int_0^\pi \int_0^{2\pi} \sin^3\varphi\sin^2\theta\, d\theta\, d\varphi = \int_0^\pi \sin\varphi(1-\cos^2\varphi)\,d\varphi \int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\, d\theta$$

$$= \left(-\cos\varphi + \frac{\cos^3\varphi}{3}\right)\Bigg|_0^\pi \cdot \frac{1}{2}\left(\theta - \cancel{\frac{\sin(2\theta)}{2}}^{\,0}\right)\Bigg|_0^{2\pi}$$

$$= \boxed{\frac{4}{3}\pi}$$

Example Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle y, x, z \rangle$ and $S: z = 1 - x^2 - y^2$ and $z = 0$

$\vec{F} = \langle y, x, 1 - x^2 - y^2 \rangle \quad \vec{n} = \langle -z_x, -z_y, 1 \rangle$
$= \langle 2x, 2y, 1 \rangle$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{n}\, dA = \iint_D (2xy + 2xy + 1 - x^2 - y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (4r^2\cancel{\cos\theta\sin\theta}^{\,0} + 1 - r^2)\, r\,dr\,d\theta$$

$$= 2\pi \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1 = \boxed{\frac{\pi}{2}}$$