Section 16.7 - Surface Integrals of Functions

MVC

- Applications:

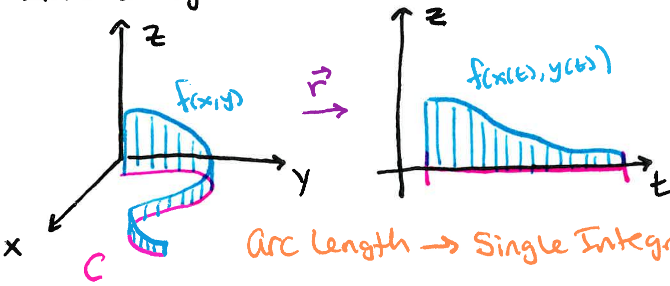
  ① Surface Area

  ② Surface Mass

  ③ Center of Mass

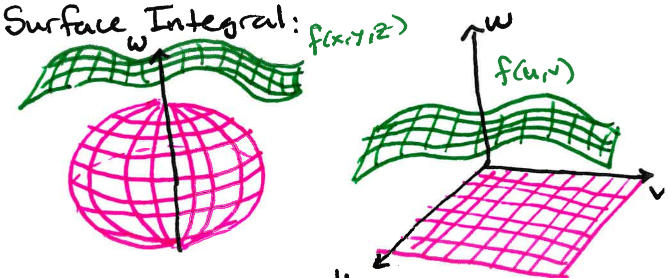
- Idea:
  - Compute Volume under $w=f(x,y,z)$ over a surface S in $\mathbb{R}^3$
  - Parametrize S to bring $w=f(x,y,z)$ over S to $\mathbb{R}^3$

Line Integral:



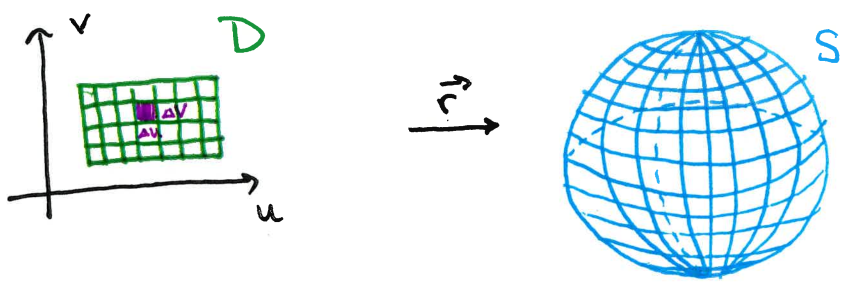
Arc Length → Single Integral

Surface Integral:



Surface Area → double integral

Computing Surface Integrals: S parametrized by $\vec{r}(u,v)=\langle x,y,z\rangle$ for $(u,v)\in D$



Recall:

$$\Delta S = |\vec{r}_u \times \vec{r}_v|\,\Delta u \Delta v$$

Volume of rectangular Prism over S:

Height × Area of Base

$$= f(x,y,z) \times \Delta S = f(r(u,v))\,|\vec{r}_u \times \vec{r}_v|\,\Delta u \Delta v$$

$(u,v)\in D$

Surface Integral of f over S:

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(r(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA}$$

Section 16.7 - Surface Integrals of functions

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$\boxed{\text{Example}}$ Compute $\iint_S x^2 \, dS$ where S is the unit sphere.

$S: \vec{r}(\theta, \varphi) = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle \quad 0 \le \theta \le 2\pi \quad 0 \le \varphi \le \pi$

$\vec{r}_\varphi = \langle \cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi \rangle \quad \vec{r}_\theta = \langle -\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0 \rangle$

$|\vec{r}_\varphi \times \vec{r}_\theta| = |\langle -\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, \sin\varphi\cos\varphi \rangle| = \sin\varphi$

$$\iint_S x^2 \, dS = \iint_D x^2 \cdot \sin\varphi \, dA = \int_0^{2\pi}\int_0^{\pi} \sin^3\varphi \cos^2\theta \, d\varphi \, d\theta$$

$$= \int_0^{\pi} \sin\varphi(1-\cos^2\varphi)\,d\varphi \cdot \frac{1}{2}\int_0^{2\pi} (1+\cos 2\theta)\,d\theta$$

$$= \left(-\cos\varphi + \frac{1}{3}\cos^3\varphi\right)\Big|_0^{\pi} \cdot \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi}$$

$$= \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) \cdot \frac{1}{2}(2\pi) = \boxed{\frac{4\pi}{3}}$$

- Application: Surface S a thin sheet with density ρ

Mass of S: $\quad m = \iint_S \rho \, dS$

Center of mass of S: $\quad (\bar{x}, \bar{y}, \bar{z}) = \frac{1}{m}\left(\iint_S x\rho \, dS, \iint_S y\rho \, dS, \iint_S z\rho \, dS\right)$

- Surface Integrals of graphs: S given by $z = g(x,y)$

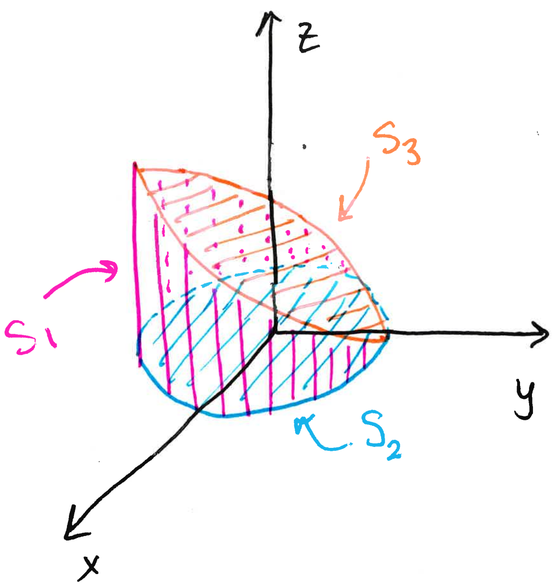
$\vec{r} = \langle x, y, g(x,y) \rangle$

$\vec{r}_x = \langle 1, 0, g_x \rangle \quad |\vec{r}_x \times \vec{r}_y| = |\langle -g_x, -g_y, 1 \rangle| = \sqrt{g_x^2 + g_y^2 + 1}$

$\vec{r}_y = \langle 0, 1, g_y \rangle$

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2 + g_y^2 + 1}\,dA$$

Example Evaluate $\iint_S z\,dS$ where S is the surface whose sides S_1 is given by $x^2+y^2=1$, base S_2 is $x^2+y^2\le 1$ in the plane $z=0$, and top S_3 is the plane $z=1+x$ above S_2.



$$\iint_S z\,ds = \iint_{S_1} z\,ds + \iint_{S_2} z\,ds + \iint_{S_3} z\,ds$$

S_1: $\langle \cos\theta, \sin\theta, z\rangle$ $\quad |\vec{r}_\theta \times \vec{r}_z| = |\langle \cos\theta, \sin\theta, 0\rangle| = 1$

$0 \le \theta \le 2\pi \qquad 0 \le z \le 1+\cos\theta$

$$\iint_{S_1} z\,ds = \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1}{2}(1+\cos(2\theta))\right) d\theta$$

$$= \frac{1}{2}\left[\theta + 2\cancel{\sin\theta}^{\,0} + \frac{1}{2}\left(\theta + \cancel{\frac{\sin(2\theta)}{2}}^{\,0}\right)\right]\Bigg|_0^{2\pi}$$

$$= \boxed{\frac{3\pi}{2}}$$

S_2: $\langle r\cos\theta, r\sin\theta, 0\rangle$

$$\iint_{S_2} z\,ds = \iint_{S_2} 0\,ds = \boxed{0}$$

S_3: $\langle x, y, 1+x\rangle = \langle r\cos\theta, r\sin\theta, 1+r\cos\theta\rangle$ $\quad |\vec{r}_r \times \vec{r}_\theta| = |\vec{r}_x \times \vec{r}_y|\cdot r$

$0 \le r \le 1 \qquad 0 \le \theta \le 2\pi \qquad = |\langle 1, 0, 1\rangle| r = \sqrt{2}r$

$$\iint_{S_3} z\,ds = \int_0^1\int_0^{2\pi} (1+r\cos\theta)\sqrt{2}r\,d\theta\,dr$$

$$= \int_0^1 2\sqrt{2}\pi r\,dr = \boxed{\sqrt{2}\pi}$$

$$\boxed{\iint_S z\,ds = \frac{3\pi}{2} + 0 + \sqrt{2}\pi}$$