Section 16.6 - Parametric Surfaces & Their Areas

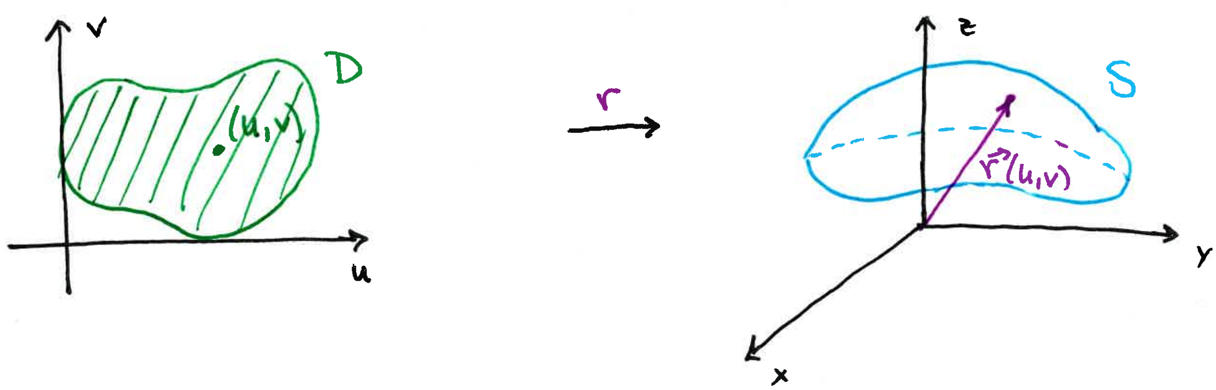
Chapter 12 - looked at special surfaces: Cylinders & Quartic Surfaces

Chapter 14 - looked at surfaces from: Functions $z = f(x,y)$

Want to describe more surfaces → Parametric Surfaces

Chapter 13 - looked at: Space curves described by vector functions $\vec{r}(t)$ with one parameter.

- <u>Parametric Surface</u>: S described by $\vec{r}(u,v) = \langle X(u,v), Y(u,v), Z(u,v) \rangle$



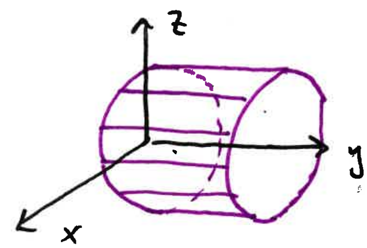
Example: Identify and sketch the surface with vector equation:

$$\vec{r}(u,v) = \langle 2\cos u, v, 2\sin u \rangle$$

Parametric Equations: $x = 2\cos u \quad y = v \quad z = 2\sin u$

relationships: $x^2 + z^2 = 4 \quad y \in \mathbb{R}$

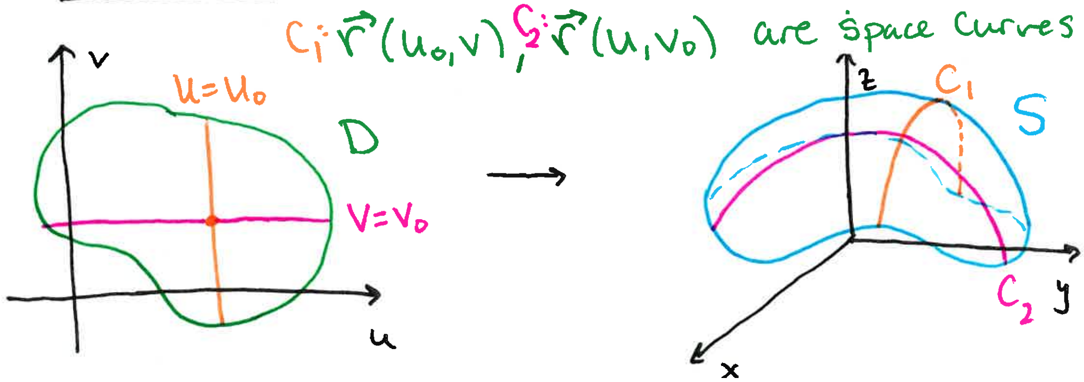
Gives circular cylinder about y-axis



- Useful Family of Curves:

<u>Grid Curves</u> - Curves where u or v is held constant

$C_1: \vec{r}(u_0,v)$, $C_2: \vec{r}(u,v_0)$ are space curves



Computers use these curves to graph in 3D

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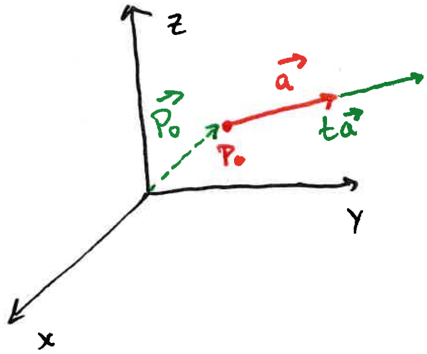
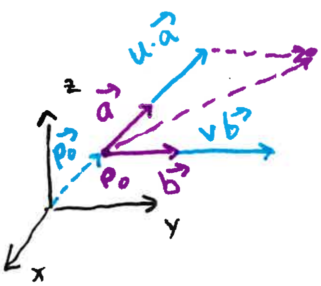
Recall: Parametrization of a line with point r_0 and vector $\vec{a}$



$$\boxed{\vec{r}(t) = \vec{P_0} + t\vec{a}}$$

Example Find a vector function that represents the plane through the point P_0, containing two non parallel vectors $\vec{a}$ and $\vec{b}$

$$\boxed{\vec{r}(u,v) = \vec{P_0} + u\vec{a} + v\vec{b}}$$



Parametric Equations:

$x = x_0 + ua_1 + vb_1 \quad y = y_0 + ua_2 + vb_2 \quad z = z_0 + ua_3 + vb_3$

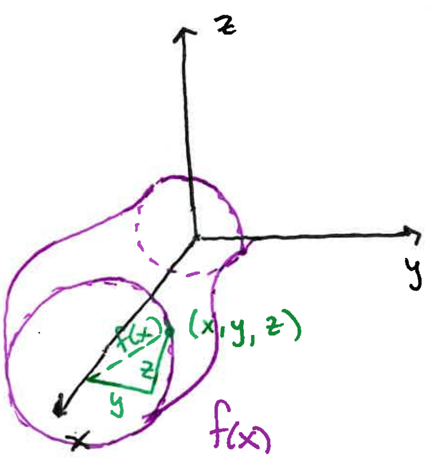
Example Find a parametric representation for the surface $z = 2\sqrt{x^2+y^2}$, that is the top half of the cone $z^2 = 4x^2 + 4y^2$.

① $x = u$, $y = v$, $z = 2\sqrt{u^2+v^2}$ with $(u,v) \in \mathbb{R}^2$

② $x = r\cos\theta$, $y = r\sin\theta$, $z = 2r$ with $r \geq 0$ $0 \leq \theta \leq 2\pi$

- Surfaces of Revolution:

Given $y = f(x)$ rotate about x-axis forms a surface



Point on Surface (x, y, z):

$$\boxed{\begin{aligned} x &= x \\ y &= f(x)\cos\theta \\ z &= f(x)\sin\theta \end{aligned}}$$

This changes the function/ if the axis changes

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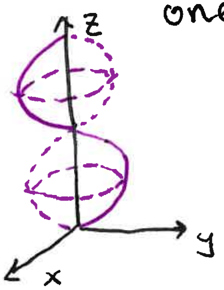
Example Find a parametrization for the surface obtained by rotating one period of $y = \sin(z)$ about the z-axis.



Parametric Equations:

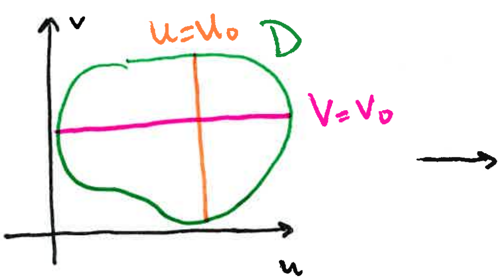
$z = z \qquad y = f(z) \cdot \cos\theta \qquad x = f(z)\sin\theta$

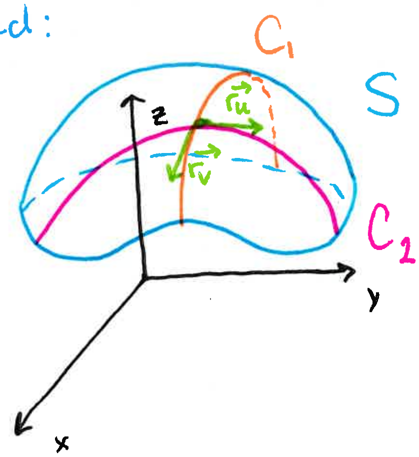
$\boxed{\begin{array}{l} 0 \le z \le 2\pi \\ 0 \le \theta \le 2\pi \end{array}}$

$\boxed{z = z \qquad y = \sin z \cdot \cos\theta \qquad x = \sin z \sin\theta}$

• Tangent Planes: Given a surface S: $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

Recall: For Equation of a plane need:





For a surface given by $z = f(x,y)$:

tangent plane at (x_0, y_0, z_0): plane containing $\vec{r}_u$, $\vec{r}_v$

normal vector: $\vec{r}_u \times \vec{r}_v\,(u_0, v_0)$

tangent plane equation: $\boxed{\vec{r}_u \times \vec{r}_v\,(u_0, v_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0}$

Example Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point $(1,1,3)$.

$\vec{r}_u = \langle 2u, 0, 1 \rangle \qquad \vec{r}_v = \langle 0, 2v, 2 \rangle \qquad x = 1 \quad y = 1 \quad z = 3$

$u = 1 \quad v = 1$

$\vec{n} = \vec{r}_u \times \vec{r}_v = \langle -2v, -4u, 4uv \rangle$

$\vec{n}(1,1) = \langle -2, -4, 4 \rangle$ tangent Plane: $\boxed{0 = -2(x-1) - 4(y-1) + 4(z-3)}$

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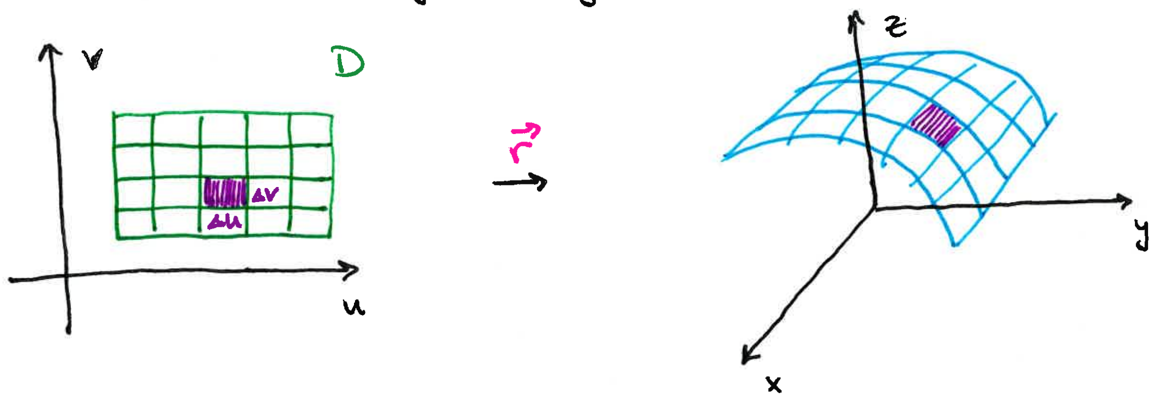
- **Surface Area:** Smooth S: $\vec{r}(u,v) = \langle x, y, z \rangle$ for $(u,v) \in D$
  Covering S only once: S smooth if $\vec{n} \neq 0$



Area of Rectangle $\approx$ $|\vec{r}_u \Delta u \times \vec{r}_v \Delta v| = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$

Surface Area of S: $A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$

Example Find the surface area of a sphere of radius a.

Parametrization: $x = a\sin\varphi\cos\theta$ $y = a\sin\varphi\sin\theta$ $z = a\cos\varphi$
$0 \leq \varphi \leq \pi$ $0 \leq \theta \leq 2\pi$

$|\vec{r}_\varphi \times \vec{r}_\theta| = a^2\sin\varphi$

$$A(S) = \iint_D a^2\sin\varphi \, dA = \int_0^{2\pi}\int_0^{\pi} a^2\sin\varphi \, d\varphi \, d\theta = \boxed{4\pi a^2}$$

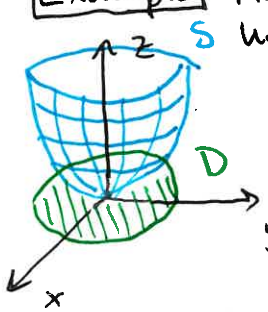
- **Surface Area of the graph of a function (Review):** $z = f(x,y)$

Parametrization: $\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$

$|\vec{r}_x \times \vec{r}_y| = |\langle -f_x, -f_y, 1 \rangle| = \sqrt{f_x^2 + f_y^2 + 1}$

$$\boxed{A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1} \, dA}$$

Example Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.



$$A(S) = \iint_D \sqrt{4x^2 + 4y^2 + 1} \, dA = \int_0^{2\pi}\int_0^{3} \sqrt{4r^2+1} \cdot r \, dr \, d\theta$$

$$= \frac{2\pi}{8} \cdot \frac{2}{3}(1 + 4r^2)^{3/2} \Big|_0^3$$

$$= \boxed{\frac{\pi}{6}\left(37^{3/2} - 1\right)}$$