Section 16.5 – Curl and Divergence

MVC

Recall: Two operations of vectors ; Two properties of a vector

① Dot Product $\vec{u} \cdot \vec{v}$ ① Magnitude of a vector

② Cross Product $\vec{u} \times \vec{v}$ ② Direction of a vector

★ Want to talk about two rates of change for vectors

① How direction of vectors change in a vector field
↳ curl of the vector field

② How the magnitude of vectors change in a vector field
↳ Divergence of the vector field

• Curl: $\vec{F} = \langle P, Q, R \rangle$ on $\mathbb{R}^3$, Partials of P, Q, R exist then:

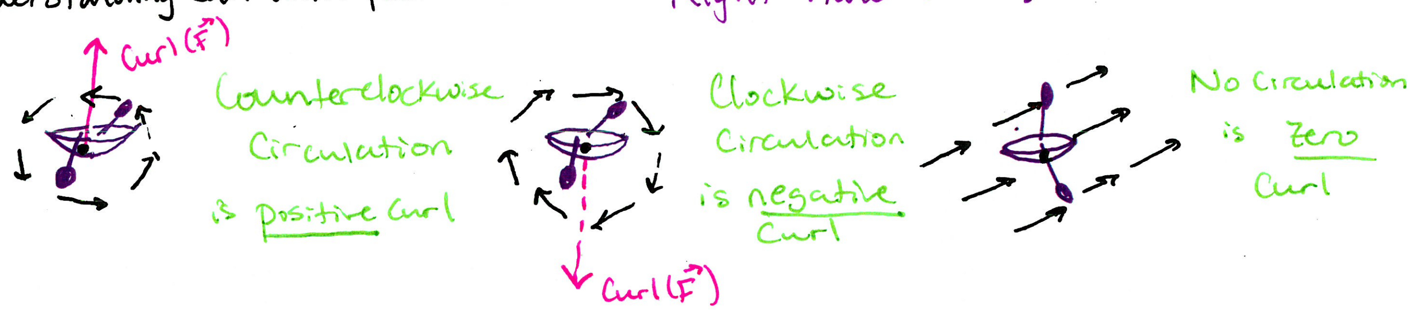
$$\boxed{\text{Curl}\,\vec{F} = \nabla \times \vec{F}} \quad \text{where } \nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

Operator Vector! No functions or values

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left\langle \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right), \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right), \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \right\rangle$$

★ Type of differentiation – result is a vector field

• Understanding Curl with paddle boats: Right Hand Rule!



★ Work on Vector field worksheet

Do they have positive, negative or zero Curl?

If Curl $\vec{F} = \vec{0}$, $\vec{F}$ is called irrotational.

Section 16.5 - Curl and Divergence

Theorem If f is a function of 3 variables with continuous second order partial derivatives then: $\text{Curl}(\nabla f) = \vec{0}$

Proof:

$$\text{Curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

By Clairaut's Theorem!

$$= \left\langle \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right), \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right), \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) \right\rangle = \vec{0} \quad \blacksquare$$

★ $\vec{F}$ Conservative $\Rightarrow$ $\text{Curl}(\vec{F}) = \vec{0}$

Theorem (Partial converse to above statement)

$\vec{F}$ defined on all $\mathbb{R}^3$, components have continuous first partials and $\text{Curl}(\vec{F}) = \vec{0}$ then $\vec{F}$ is conservative.

Example (a) Show $\vec{F} = \langle y^2 z^3, 2xyz^3, 3xy^2z^2 \rangle$ is conservative.

(b) Find f so that $\nabla f = \vec{F}$.

(a) $\text{Curl}(\vec{F}) = \langle (6xyz^2 - 6xyz^2), (3y^2z^2 - 3y^2z^2), (2yz^3 - 2yz^3) \rangle = \vec{0}$

Since $\vec{F}$ is defined on all $\mathbb{R}^3$ $\Rightarrow$ $\vec{F}$ is conservative. ✓

(b)

$$\left.\begin{aligned} f_x &= y^2z^3 \Rightarrow f = \int y^2z^3\,dx = \underline{y^2z^3x} + g_1(y,z) \\ f_y &= 2xyz^3 \Rightarrow f = \int 2xyz^3\,dy = \underline{y^2z^3x} + g_2(x,z) \\ f_z &= 3xy^2z^2 \Rightarrow f = \int 3xy^2z^2\,dz = \underline{y^2z^3x} + g_3(x,y) \end{aligned}\right\} \boxed{f(x,y,z) = y^2z^3x}$$

• Green's Theorem Rewritten: $\displaystyle\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$

$\vec{F} = \langle P, Q, 0 \rangle$ $\quad \text{Curl}(\vec{F}) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_D \text{Curl}(\vec{F})\,dA}$$

Section 16.5 - Curl and Divergence

- Divergence: $\vec{F} = \langle P, Q, R \rangle$ defined on $\mathbb{R}^3$, Partials of P, Q, R exist then:

$$\boxed{\text{div}(\vec{F}) = \nabla \cdot \vec{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad \text{where } \nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

- Understanding Divergence:

★ Moving away from a point in direction of vectors

Do the vector's magnitude increase ($\text{div}(\vec{F}) > 0$), decrease ($\text{div}\vec{F} < 0$) or no change ($\text{div}\vec{F} = 0$)

★ Work on Vector field worksheet

If $\text{div}\vec{F} = 0$, $\vec{F}$ is called incompressible

Do they have positive, negative or zero divergence?

$\boxed{\text{Theorem}}$ $\vec{F} = \langle P, Q, R \rangle$ on $\mathbb{R}^3$, P, Q, R have continuous second partials then

$$\text{Div}(\text{curl}\,\vec{F}) = \boxed{0}$$

Proof: $\text{div}(\text{curl}\,\vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = (\nabla \times \nabla) \cdot \vec{F} = \vec{0} \cdot \vec{F} = 0$ ∎

Property of Cross product!

$\boxed{\text{Example}}$ Show $\vec{F} = \langle xz, xyz, -y^2 \rangle$ can't be written as the curl of another vector field, that is $\vec{F} \neq \text{Curl}\,\vec{G}$ for any $\vec{G}$.

$\text{div}(\vec{F}) = z + xy + 0 \neq 0$ for all $\mathbb{R}^3$

But if $\vec{F} = \text{Curl}\,\vec{G}$ then by theorem above $\text{div}\,\vec{F} = 0$. ∎

- Laplace Operator:

$$\text{div}(\nabla f) = \nabla \cdot (\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplace Equation: $\nabla^2 f = 0$

★ Watch water flow video on website