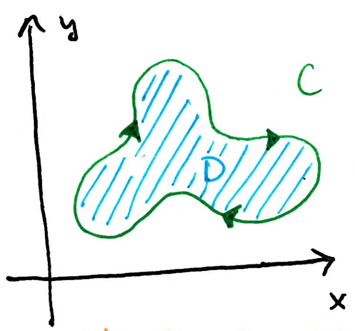
Section 16.4 - Green's Theorem

For Conservative vector fields have FTC for Line integrals

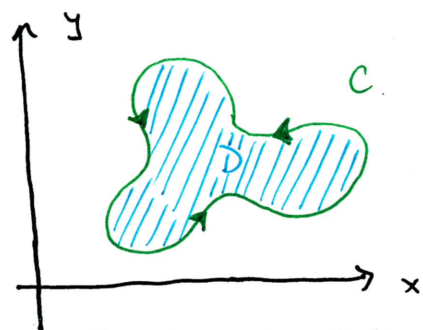
★ Now want something for non-Conservative vector fields

- Positive Orientation:

  For simple closed curves positive orientation refers to a single counterclockwise traversal of C.



Negative Orientation



Positive Orientation

- Notation: C a simple closed curve

$$\int_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot d\vec{r}$$

$$\int_{-C} \vec{F}\cdot d\vec{r} = \oint_{-C} \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot d\vec{r}$$

∂D means boundary of D

Green's Theorem Let C be a positively oriented, piecewise smooth, simple closed curve. C bounds D i.e. $\partial D = C$. P, Q have continuous first order partials on D

$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA}$$

Recall: $\vec{F} = \langle P, Q\rangle$ $\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy$

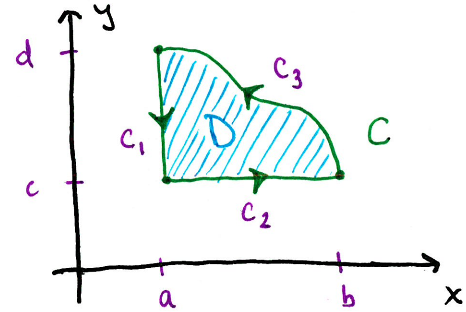
$\vec{F}$ conservative $\Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow \int_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_D 0\,dA = 0$

on closed curves.

Section 16.4 - Green's Theorem

• **Proof:** Any Curve C piecewise smooth, simple closed can be broken into rectangles or as follows:

$D = \{(x,y) \mid a \le x \le b \quad c \le y \le f(x)\}$
$\quad = \{(x,y) \mid c \le y \le d \quad a \le x \le g(y)\}$
where f, g are inverses



$C = C_1 \cup C_2 \cup C_3$

Show: ① $\displaystyle\int_C P\,dx = \iint_D -\frac{\partial P}{\partial y}\,dA$

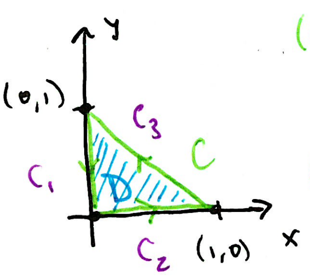
② $\displaystyle\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$

① $$\int_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{C_3} P\,dx = \int_a^a \overset{0}{\cancel{P\,dx}} + \int_a^b P(x,c)\,dx + \int_b^a P(x,f(x))\,dx$$
$$= 0 + \int_a^b P(x,c) - P(x,f(x))\,dx = \int_a^b \int_{f(x)}^c \frac{\partial P}{\partial y}\,dy\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA \checkmark$$

② $$\int_C Q\,dy = \int_{C_1} Q\,dy + \int_{C_2} Q\,dy + \int_{C_3} Q\,dy = \int_d^c Q(a,y)\,dy + \overset{0}{\cancel{\int_c^c Q\,dy}} + \int_c^d Q(g(y),y)\,dy$$
$$= \int_c^d Q(g(y),y) - Q(a,y)\,dy = \int_c^d \int_a^{g(y)} \frac{\partial Q}{\partial x}\,dx\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA \checkmark$$

∎

Example Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve from (0,0) to (1,0) to (0,1) using (a) Green's theorem and (b) Line Integrals.



(a) $$\int_C x^4\,dx + xy\,dy = \iint_D (y-0)\,dA = \int_0^1 \int_0^{1-x} y\,dy\,dx = \int_0^1 \tfrac{1}{2}(1-x)^2\,dx = \boxed{\tfrac{1}{6}}$$

(b) $$\int_C x^4\,dx + xy\,dy = \int_{C_1} \underbrace{\cancel{x^4\,dx + xy\,dy}}_{0 \text{ as } x=0} + \int_{C_2} x^4\,dx + \underbrace{\cancel{xy\,dy}}_{0 \text{ as } y=0} + \int_{C_3} x^4\,dx + xy\,dy \quad y=1-x \quad dy=-dx$$
$$= \int_0^1 x^4\,dx - \int_0^1 x^4 - x(1-x)\,dx$$
$$= \frac{1}{5} - \left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right) = \boxed{\frac{1}{6}}$$

Section 16.4 - Green's Theorem

- Reverse Application of Green's theorem:

$$A = \text{Area of } D = \iint_D 1\, dA \quad \text{write as line integrals.}$$

Find P, Q so that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

① $P = 0$, $Q = x$ ② $P = -y$, $Q = 0$ ③ $P = -\frac{1}{2}y$, $Q = \frac{1}{2}x$

$$\boxed{A = \oint_{\partial D} x\,dy = \oint_{\partial D} -y\,dx = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx}$$

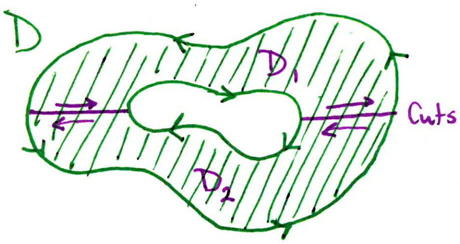
Example Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$x = a\cos t \quad y = b\sin t \quad 0 \le t \le 2\pi$

$$A = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,dt = \boxed{\pi ab}$$

- Green's Theorem on Regions with holes:



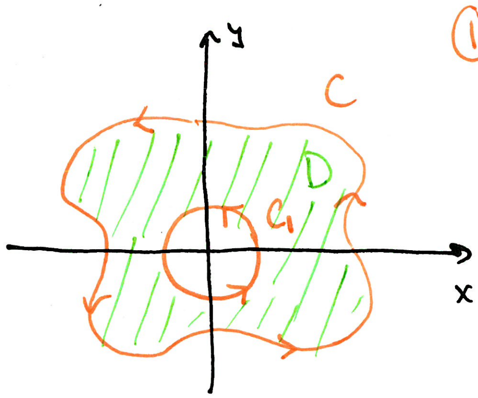
$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA + \iint_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$= \int_{\partial D_1} P\,dx + Q\,dy + \int_{\partial D_2} P\,dx + Q\,dy$$

Cuts subtract out

$$= \int_{\partial D} P\,dx + Q\,dy \quad \checkmark$$

Example $\vec{F}(x,y) = \frac{\langle -y, x\rangle}{|\langle -y, x\rangle|^2}$ show $\int_C \vec{F}\cdot d\vec{r} = 2\pi$ for every positively oriented simple closed path around the origin.



① Show any simple closed path C around origin yields same line integral as the unit circle path C_1

$$\int_C \vec{F}\cdot d\vec{r} + \int_{-C_1} \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{(x^2+y^2) - 2x^2}{|\langle -y,x\rangle|^4} - \left(\frac{(-1)(x^2+y^2) + 2y^2}{|\langle -y,x\rangle|^4}\right) = 0$$

$$\Rightarrow \boxed{\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r}} \quad \checkmark$$

② $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} -\sin t(-\sin t) + \cos t(\cos t)\,dt = \boxed{2\pi}$

Section 16.4 - Green's Theorem

MVC

Recall: $\boxed{\text{Theorem (16.3)}}$ $\vec{F} = \langle P, Q \rangle$ on open simply-connected region D. P, Q have continuous first order partial Derivatives with $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D then $\vec{F}$ is Conservative.

• Proof:

C simple closed curve in D with region R bounded by C

By Green's Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = 0$$

★ Any closed curve can be broken into simple closed curves.

So $\int_C \vec{F} \cdot d\vec{r} = 0$ for all closed curves C in D

$\Rightarrow \int_C \vec{F} \cdot d\vec{r}$ is path independent

$\Rightarrow$ By FTC for Line Integrals $\vec{F}$ is Conservative ■