Section 16.3 - Fundamental Theorem for Line Integrals

MVC

- **Recap:** <u>Part 1:</u> $\int_C \nabla f \cdot d\vec{r} = f(r(b)) - f(r(a))$

  where $C: \vec{r}(t),\ a \le t \le b$ is piecewise smooth

  f differentiable, ∇f continuous on C

  <u>Part 2:</u> $\int_C \vec{F} \cdot d\vec{r}$ independent of Path on $D \Rightarrow \vec{F}$ Conservative on D

  for $\vec{F}$ continuous on Open connected D

- **Goal:** Showing $\int_C \vec{F} \cdot d\vec{r}$ is independent of path is hard

  <u>Want</u> an easier way to Show $\vec{F}$ is Conservative!

- <u>Idea:</u> Assume $\vec{F}$ is conservative work backwards to find conditions on $\vec{F}$.

  Suppose $\vec{F} = \langle P, Q \rangle$ is a conservative vector field

  that means: $\nabla f = \vec{F}$ so $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$

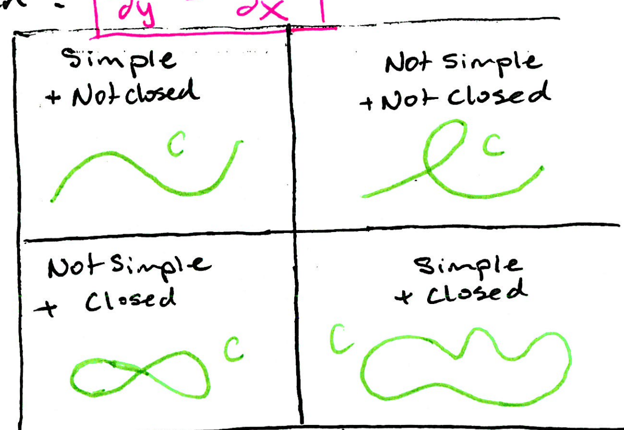
  Assume P, Q have continous first order partial Derivatives

  So by: <u>Clairaut's Theorem</u> $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \Rightarrow \boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$

Theorem If $\vec{F}(x,y) = \langle P, Q \rangle$ is Conservative where P, Q have continuous first order partial derivatives then: $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$

Want a Converse of this!

★ Need special regions



- Simply-Connected Regions:

  <u>Simple Curve</u>:

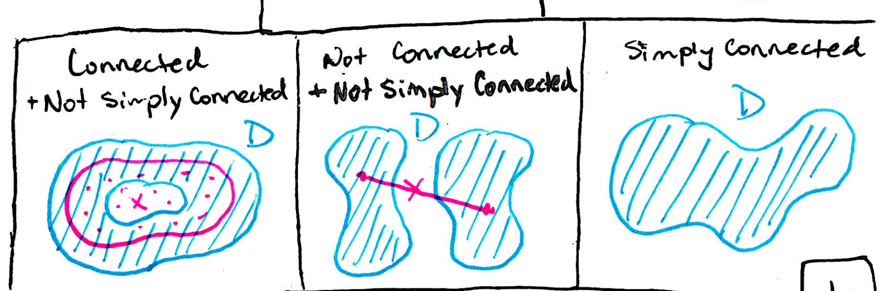
  A curve that doesn't intersect itself.

  <u>Closed Curve</u>:

  Curve with $\vec{r}(b) = \vec{r}(a)$

  <u>Simply-Connected region</u>:

  Connected region D with all simple closed curves in D enclosing points only in D.



Theorem (Partial Converse of the Last theorem)

$\vec{F} = \langle P, Q \rangle$ a vector field on an open simply connected region D, P, Q have continuous first order partial Derivatives with

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D then: $\vec{F}$ is conservative.

Proof: Consequence of Green's Theorem next section 16.4.

Example Determine if $\vec{F}(x,y) = \langle 3 + 2xy, x^2 - 3y^2 \rangle$ is conservative. If it is find its potential function.

$P(x,y) = 3 + 2xy \qquad Q(x,y) = x^2 - 3y^2$

$\frac{\partial P}{\partial y} = 2x \qquad \frac{\partial Q}{\partial x} = 2x \qquad$ since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $D = \mathbb{R}^2$ open simply connected

$\Rightarrow \vec{F}$ is conservative

$P = \frac{\partial f}{\partial x} \Rightarrow f(x,y) = \int P\,dx = 3x + x^2y + g(y)$

$Q = \frac{\partial f}{\partial y} \Rightarrow f(x,y) = \int Q\,dy = x^2y - y^3 + h(x)$

Thus a potential function for $\vec{F}$ is $\boxed{f(x,y) = x^2y - y^3 + 3x}$

Example Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $C: \vec{r}(t) = \langle e^t \sin t, e^t \cos t \rangle$ $0 \le t \le \pi$ and $\vec{F}(x,y) = \langle 3 + 2xy, x^2 - 3y^2 \rangle$.

By above example $\vec{F} = \nabla f$ where $f(x,y) = x^2y - y^3 + 3x$

So $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(r(\pi)) - f(r(0))$

$= f(0, -e^{\pi}) - f(0, 1)$

$= \boxed{e^{3\pi} + 1}$

- Conservation of Energy:

$\vec{F}$ a force field moves an object of mass m along a curve $C: \vec{r}(t)$ $a \le t \le b$

Newton's Second Law: $\vec{F}(r(t)) = m \times \vec{r}''(t)$

force = mass x acceleration

Work done $= \int_C \vec{F} \cdot d\vec{r} = \int_a^b m\, \vec{r}''(t) \cdot \vec{r}'(t)\, dt$

Recall: $\frac{d}{dt}\left(\vec{r}'(t) \cdot \vec{r}'(t)\right) = 2\vec{r}''(t) \cdot \vec{r}'(t)$

$= \int_a^b m \frac{1}{2} \frac{d}{dt}\left(\vec{r}'(t) \cdot \vec{r}'(t)\right) dt = \frac{m}{2}\int_a^b \frac{d}{dt}|\vec{r}'(t)|^2 dt$

$= \frac{m}{2}|\vec{r}'(b)|^2 - \frac{m}{2}|\vec{r}'(a)|^2 = K(b) - K(a)$

Kinetic Energy of the object $K(t) = \frac{1}{2}m|v(t)|^2 = \frac{1}{2}m|\vec{r}'(t)|^2$

Assume $\vec{F}$ is conservative so: $\vec{F} = \nabla f$

Potential Energy of the object at (x,y,z) is defined by:

$$P(x,y,z) = -f(x,y,z) \Rightarrow \vec{F} = -\nabla P$$

By Fundamental Theorem for Line integrals we have:

$$\text{Work done} = \int_C \vec{F} \cdot d\vec{r} = \int_C -\nabla P \cdot d\vec{r} = P(r(a)) - P(r(b))$$

Law of Conservation of Energy:

$$K(b) - K(a) = P(r(a)) - P(r(b))$$

$$\Downarrow$$

$$K(b) + P(r(b)) = K(a) + P(r(a))$$

Sum of Potential and Kinetic Energy remains constant.