Section 16.3 - Fundamental Theorem for Line Integrals

MVC

Recall: Fundamental Theorem of Calculus (FTC)

Part 1: F' continuous on $[a,b]$, $\int_a^b F'(x) = F(b) - F(a)$

Part 2: $F(x) = \int_a^x F'(x)dx$ where $\frac{d}{dx}(F(x)) = F'(x)$

$\boxed{\text{Theorem}}$ FTC for Line Integrals Part 1:

C a smooth curve given by $\vec{r}(t)$, $a \le t \le b$, f differentiable with ∇f continuous on C then $\boxed{\int_C \nabla f \cdot d\vec{r} = f(r(b)) - f(r(a))}$

Proof: $\int_C \nabla f \cdot d\vec{r} = \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$

$$= \int_a^b \left(\frac{\partial f}{dx} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} \right) dt$$

By Chain Rule $= \int_a^b \frac{d}{dt}(f(r(t))\,dt = f(r(b)) - f(r(a))$ by FTC ∎

★ Note: The FTC for line Integrals Part 1 also holds for piecewise smooth curves.

- Conservative Vector Field:

A vector field $\vec{F}$ is <u>conservative</u> if it is the gradient field of some function f. That is $\vec{F} = \nabla f$ and we say f is a <u>potential function</u> for $\vec{F}$.

★ Recall: In general $\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}$ even if C_1, C_2 start and end at the same points. But by the theorem we have conservative vector field do Not depend on the path!

- Independence of Path:

$\vec{F}$ continuous on D, $\int_C \vec{F} \cdot d\vec{r}$ is <u>independent of path</u> if

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

for any two paths C_1, C_2 in D with the same initial & terminal points.

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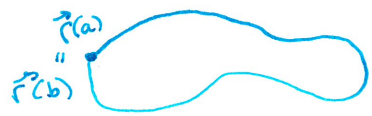
Goal: ① Want part 2 of FTC for Line Integrals - i.e. writing the potential function as a line Integral.

② But how do we know we can? Need $\vec{F}$ path independent

③ Definition of Path Independence hard to check → Find easier way

- Closed curves:



A curve C is <u>closed</u> if its terminal and initial point are the same.

$\int_C \vec{F}\cdot d\vec{r}$ independent of path in D, C closed in D then

$$\int_C \vec{F}\cdot d\vec{r} = 0$$

Theorem $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D iff

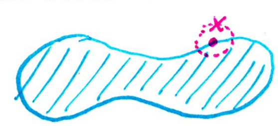
$\int_C \vec{F}\cdot d\vec{r} = 0$ for all closed paths C in D.

Open Not Connected

- Open Connected Regions:

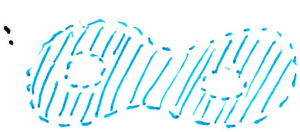
D is <u>open</u> if for all points P in D there is a disk with center P completely in D.

Connected Not Open



D is <u>Connected</u> if any two points in D can be joined with a path completely in D.

Open And Connected



Theorem Fundamental Theorem for Line Integrals Part 2:

$\vec{F}$ continuous on open connected region D.

$\int_C \vec{F}\cdot d\vec{r}$ independent of Path in D $\Rightarrow$ $\vec{F}$ conservative on D.

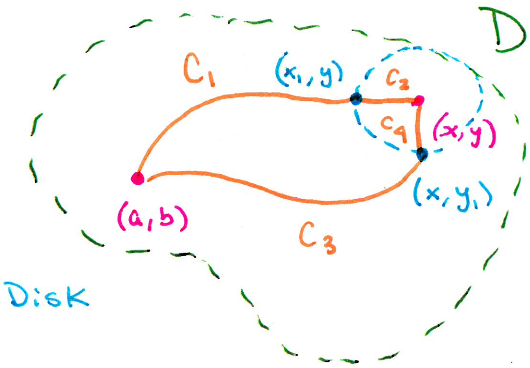
That is there is a potential function for $\vec{F}$ with $\vec{F} = \nabla f$.

Proof: FTC for Line Integrals part 2

Assume $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D

Show: There is f with $\nabla f = \vec{F}$ (means $\vec{F}$ is conservative)

Let $f(x,y) = \int_{(a,b)}^{(x,y)} \vec{F}\cdot d\vec{r}$ with (a,b) in D.



Since D is Open there is an open Disk about (x,y) in D.

Choose points (x_1,y) and (x,y_1) in the Disk
$x_1 \neq x$ and $y_1 \neq y$

Since D is simply connected there are paths $C_1: (a,b) \to (x_1,y)$ and $C_3: (a,b) \to (x,y_1)$ completely in D.

$\int_C \vec{F}\cdot d\vec{r}$ path independent means: $f(x,y) = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$ [C_1 has fixed x variable y]

Thus $\frac{\partial}{\partial x} f(x,y) = 0 + \frac{\partial}{\partial x}\int_{C_2} \vec{F}\cdot d\vec{r} = \frac{\partial}{\partial x}\int_{C_2} P\,dx + Q\,dy = P$

Likewise $f(x,y) = \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r}$ so $\quad \vec{F} = \langle P, Q\rangle = \nabla f$

$\frac{\partial}{\partial y} f(x,y) = 0 + \frac{\partial}{\partial y}\int_{C_4} \vec{F}\cdot d\vec{r} = \frac{\partial}{\partial y}\int_{C_4} P\,dx + Q\,dy = Q$ ∎

Example Let $f(x,y) = \sin(x-2y)$. Compute $\int_C \nabla f \cdot d\vec{r}$ where C is any curve that starts at $(0,0)$ and ends at $(\pi/2, \pi/2)$. Then find a curve not closed C_1 so that $\int_{C_1} \nabla f\cdot d\vec{r} = 0$.

By FTC for line Integrals: $\int_C \nabla f\cdot d\vec{r} = f(\pi/2, \pi/2) - f(0,0) = \sin(-\pi/2) = \boxed{-1}$

C_1: $(0,0)$ to $(0, \pi/2)$ then

$$\int_{C_1} \nabla f \cdot d\vec{r} = f(0,\pi/2) - f(0,0) = \sin(-\pi) = 0$$