Section 16.2 - Line Integrals in Space & of Vector Fields

- Line integral of f over C:

$C: \quad x = x(t), \quad y = y(t), \quad z = z(t), \quad a \le t \le b : \vec{r}(t)$

wrt arc length: $\displaystyle \int_C f\,ds = \int_a^b f(r(t))\,|\vec{r}\,'(t)|\,dt$

Similar wrt x, y, z $\displaystyle = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

$\boxed{\text{Example}}$ Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the lines C_1 from $(2,0,0)$ to $(3,4,5)$ followed by C_2 from $(3,4,5)$ to $(3,4,0)$.

Recall: parametrization of line segment from r_0 to r_1: $\vec{r}(t) = \vec{r}_0 + (\vec{r}_1 - \vec{r}_0)t \quad 0 \le t \le 1$

C_1: $\vec{r}_1(t) = \langle 2,0,0\rangle + \langle 1,4,5\rangle t \quad x = 2+t \quad y = 4t \quad z = 5t \quad 0 \le t \le 1$

$dx = dt \quad dy = 4dt \quad dz = 5dt$

C_2: $\vec{r}_2(t) = \langle 3,4,5\rangle + \langle 0,0,-5\rangle t \quad x = 3 \quad y = 4 \quad z = 5-5t \quad 0 \le t \le 1$

$dx = 0dt \quad dy = 0dt \quad dz = -5dt$

$$\int_C y\,dx + z\,dy + x\,dz = \int_{C_1} y\,dx + z\,dy + x\,dz + \int_{C_2} y\,dx + z\,dy + x\,dz$$

$$= \int_0^1 (4t)dt + (5t)(4dt) + (2+t)(5dt) + \int_0^1 (4)(0dt) + (5-5t)(0dt) + 3(-5dt)$$

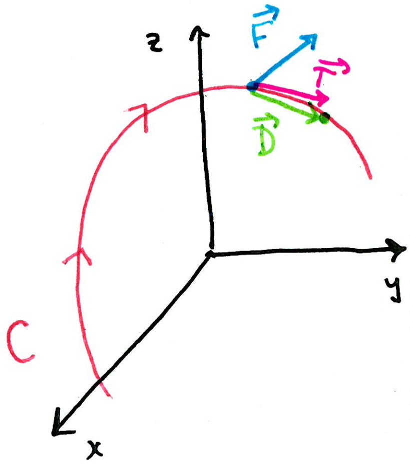
$$= \int_0^1 29t - 5\,dt = \frac{29}{2} - 5 = \boxed{19/2}$$

- Line Integrals of Vector Fields: We will understand this by way of an application
  - Work done by force $F(x)$ in the x-direction from $x=a$ to $x=b$

$$\boxed{W = \int_a^b f(x)\,dx}$$

  - Now suppose $\vec{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ is a continuous force field on $\mathbb{R}^3$

Compute work done to move a particle along curve C in $\mathbb{R}^3$.



$\Delta W = \vec{F} \cdot \vec{D}$ $\quad \vec{D} \approx \vec{T}\Delta s$ $\quad$ $\vec{T}$ unit tangent scaled by small change in arc length

$$W = \int_C \vec{F} \cdot \overbrace{\vec{T}\,ds}^{d\vec{r}} \sim \text{Understanding}$$

Recall: $\vec{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ and $ds = |\vec{r}\,'(t)|dt$

Since $s = \int_a^t |\vec{r}\,'(u)|du$

$$= \boxed{\int_a^b \vec{F}(r(t)) \cdot \vec{r}\,'(t)\,dt} \sim \text{How to Compute}$$

$$= \int_C \vec{F} \cdot d\vec{r} \sim \text{Notation}$$

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- **Definition:** $\vec{F}$ continuous on smooth $C: \vec{r}(t)$ $a \le t \le b$ then the Line integral of $\vec{F}$ over C is: $\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(r(t)) \cdot \vec{r}\,'(t)\,dt}$

- **Orientation Change:**

$$\star \int_{-C} \vec{F} \cdot d\vec{r} = \int_{-C} \vec{F} \cdot \vec{T}\,ds = \int_C \vec{F} \cdot (-\vec{T})\,ds = -\int_C \vec{F} \cdot dr$$

- **Notation:**

$\vec{F} = \langle P, Q, R \rangle$ then $\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$

Since $\int_C \vec{F} \cdot dr = \int_a^b \langle P, Q, R \rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle\,dt = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right) \cdot dt$

Example Find the work done by the force field $\vec{F}(x,y) = \langle x^2, -xy \rangle$ in moving a particle along $\vec{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le \pi/2$.

$\vec{F}(r(t)) = \langle \cos^2 t, -\cos t \cdot \sin t \rangle$ $\quad \vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \langle \cos^2 t, -\cos t \sin t \rangle \cdot \langle -\sin t, \cos t \rangle\,dt$$

$$= \int_0^{\pi/2} -\cos^2 t \cdot \sin t - \cos^2 t \cdot \sin t\,dt$$

$$= -2\,\frac{\cos^3 t}{3}(-1)\Big|_0^{\pi/2}$$

$$= \boxed{-\frac{2}{3}}$$

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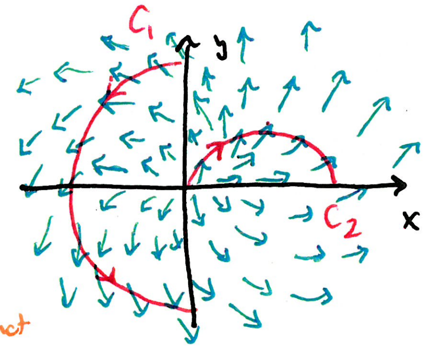
- Extra Examples

#18. Are the line integrals of $\vec{F}$ over C_1 and C_2 positive, negative or zero? Explain.



Recall: for nonzero $\vec{u},\vec{v}$, $\vec{u}\cdot\vec{v}=0$ iff $\vec{u}\perp\vec{v}$.

Vectors going in direction of C_1/C_2 yield a positive dot product and in opposite direction yield negative dot product

so $\int_{C_1}\vec{F}\cdot d\vec{r} > 0$ and $\int_{C_2}\vec{F}\cdot d\vec{r} \le 0$

#21. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}=\langle \sin x, \cos y, xz\rangle$ $\vec{r}(t)=\langle t^3, -t^2, t\rangle$ $0\le t\le 1$

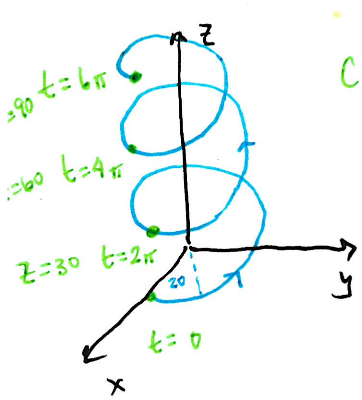
$\vec{F}(r(t))=\langle \sin(t^3), \cos(-t^2), t^4\rangle \qquad \vec{r}\,'(t)=\langle 3t^2, -2t, 1\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 3t^2\cdot\sin(t^3) - 2t\cdot\cos(-t^2) + t^4\,dt$$

$$= -\cos(t^3) + \sin(-t^2) + \frac{t^5}{5}\Big|_0^1$$

$$= \boxed{-\cos(1) + \sin(-1) + \frac{1}{5} + 1}$$

#45. A 160-lb man carries 25-lb can of paint up a helical staircase that encircles a silo with a radius of 20 ft. If the silo is 90 ft tall and the man makes exactly 3 revolutions climbing to the top, find the work done by the man against gravity.



$C: \vec{r}(t)=\langle 20\cos t, 20\sin t, \frac{15}{\pi}t\rangle \qquad 0\le t\le 6\pi$

$\vec{F}=\langle 0, 0, 160+25\rangle$ lb $\qquad \vec{r}\,'(t)=\langle -20\sin t, 20\cos t, \frac{15}{\pi}\rangle$

$$W=\int_C \vec{F}\cdot d\vec{r} = \int_0^{6\pi} 185\cdot\frac{15}{\pi}\,dt = \boxed{185\cdot 15\cdot 6 \text{ ft}\cdot\text{lbs}}$$