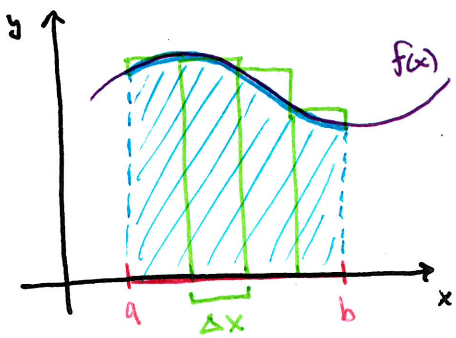
Section 16.2 - Line Integrals of Functions

MVC

Goal: Integrate vector fields but before we do must understand Integration of functions first.

★ Ribbon of paper activity

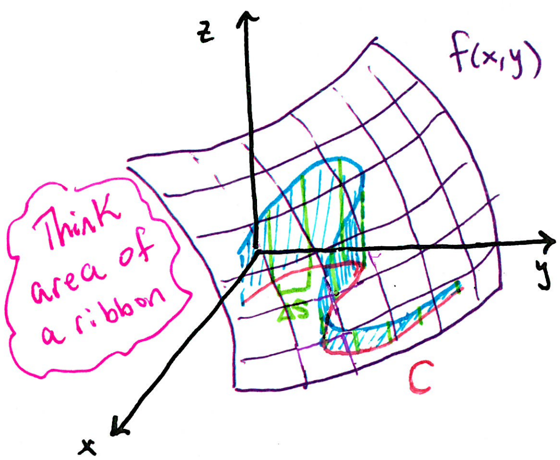
- Line Integral in 2D



Area of a rectangle $= f(x) \cdot \Delta x$

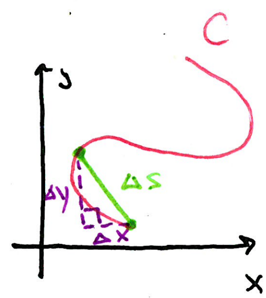
$$\text{Area} = \int_a^b f(x)\,dx$$

- Line Integral in 3D



Area of rectangle $= f(x,y) \cdot \Delta s$

where Δs is a small change in arc length of C



$$\text{Area} = \int_C f(x,y)\,dS$$

$$\Delta S = \sqrt{\Delta x^2 + \Delta y^2}$$

$$C: \vec{r}(t) = \langle x(t), y(t) \rangle$$

$$\Delta x = x'(t)\,\Delta t \qquad \Delta y = y'(t)\,\Delta t$$

So $\Delta S = \sqrt{(x'(t))^2 + (y'(t))^2}\,\Delta t$

- Line integral for f above C wrt arc length:

f defined on smooth curve C

$C: \vec{r}(t) = \langle x(t), y(t) \rangle \quad a \le t \le b$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_a^b f(r(t))\,|\vec{r}\,'(t)|\,dt$$

Area of Ribbon

- Other Line integrals for f above C:
  - Wrt x: $\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t)) \cdot \left(\frac{dx}{dt}\right) dt$

    Area you see when looking towards x-axis
  - Wrt y: $\int_C f(x,y)\,dy = \int_a^b f(x(t), y(t)) \cdot \left(\frac{dy}{dt}\right) dt$

    Area you see when looking towards y-axis

★ Visit Line integral demo on Website

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- Changing Direction: $-C$ means travel C backwards

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \qquad \star \int_{-C} f\,ds = \int_C f\,ds$$

arc length doesn't change sign based on direction of C!

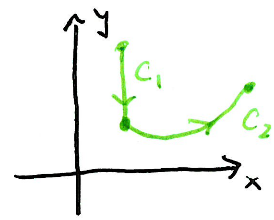
$$\int_{-C} f\,dx = -\int_C f\,dx \qquad \int_{-C} f\,dy = -\int_C f\,dy$$

- Properties:

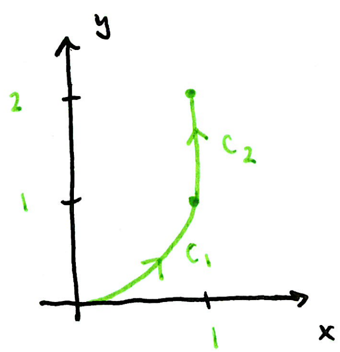
Recall: $a<c<b$ then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Similar: C piecewise-smooth union $C = C_1 \cup C_2$ then

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$$



$\boxed{\text{Example}}$ Evaluate $\int_C 2x\,ds$ where C consists of C_1: $y=x^2$ from $(0,0)$ to $(1,1)$ and C_2: vertical line from $(1,1)$ to $(1,2)$.



C_1: $x=t \quad y=t^2 \quad 0\le t\le 1$

$\frac{dx}{dt}=1 \quad \frac{dy}{dt}=2t \quad ds=\sqrt{1+4t^2}\,dt$

C_2: $x=1 \quad y=t \quad 1\le t\le 2$

$\frac{dx}{dt}=0 \quad \frac{dy}{dt}=1 \quad ds=\sqrt{1}\,dt$

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$$

$$= \int_0^1 2(t)\sqrt{1+4t^2}\,dt + \int_1^2 2(1)\,dt$$

$$= \frac{\frac{2}{3}(1+4t^2)^{\frac{3}{2}}}{4}\Bigg|_0^1 + 2 = \boxed{\frac{1}{6}\left(5^{3/2}-1\right)+2}$$

- Notation:

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

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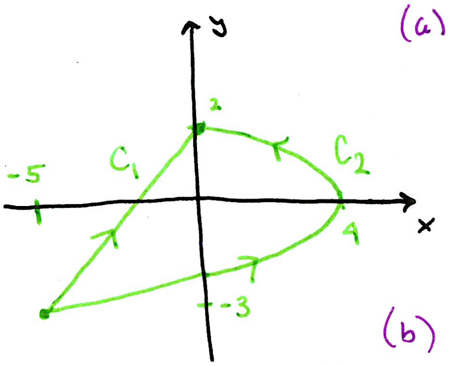
Example Evaluate $\int_C y^2\,dx + x\,dy$, where (a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$ and (b) $C = C_2$ is the arc $x = 4-y^2$ from $(-5,-3)$ to $(0,2)$.



(a) C_1: $x = t \quad y = t+2 \quad -5 \le t \le 0$

$dx = dt \quad dy = dt$

$$\int_C y^2\,dx + x\,dy = \int_{-5}^{0} (t+2)^2\,dt + (t)\,dt = \frac{(t+2)^3}{3} + \frac{t^2}{2}\Big|_{-5}^{0}$$

$$= \frac{8}{3} - \frac{-27}{3} - \frac{25}{2} = \boxed{-\frac{5}{6}}$$

(b) C_2: $x = 4-t^2 \quad y = t \quad -3 \le y \le 2$

$dx = -2t\,dt \quad dy = dt$

$$\int_C y^2\,dx + x\,dy = \int_{-3}^{2} t^2(-2t)\,dt + (4-t^2)\,dt = \frac{-2t^4}{4} + 4t - \frac{t^3}{3}\Big|_{-3}^{2}$$

$$= -8 + 8 - \frac{8}{3} + \frac{81}{2} + 12 - 9 = \boxed{\frac{245}{6}}$$

★ Conclusion: In general, line integrals are dependent on the path!

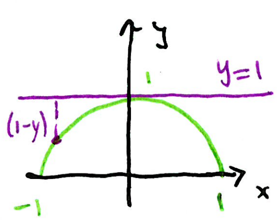
- Application of Line Integrals:

$f(x,y) = \rho(x,y)$ density function of a thin wire

Mass of wire: $m = \int_C \rho(x,y)\,ds$

Center of mass: $(\bar{x}, \bar{y}) \quad \bar{x} = \frac{1}{m}\int_C x\rho(x,y)\,ds \quad \bar{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds$

Example A wire is in the shape of the semicircle $x^2+y^2=1$, $y \ge 0$ and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from $y=1$.



C: $x = \cos t \quad y = \sin t \quad 0 \le t \le \pi \qquad \rho(x,y) = K(1-y)$

$\frac{dx}{dt} = -\sin t \quad \frac{dy}{dt} = \cos t \quad ds = dt$

$$m = \int_0^{\pi} K(1-\sin t)\,dt = K(t+\cos t)\Big|_0^{\pi} = K(\pi-2)$$

By symmetry $\bar{x} = 0$

$$\bar{y} = \frac{1}{K(\pi-2)}\int_0^{\pi} (\sin t)(K(1-\sin t))\,dt = \frac{1}{\pi-2}\int_0^{\pi} \sin t - \left(\frac{1-\cos(2t)}{2}\right)dt$$

$$= \frac{1}{\pi-2}\left(-\cos t - \frac{1}{2}\left(t - \cancel{\frac{\sin(2t)}{2}}^{\,0}\right)\right)\Big|_0^{\pi} = \boxed{\frac{1}{\pi-2}\left(2-\frac{1}{2}\pi\right)} \approx 0.376$$