Section 16.1 - Vector Fields

Chapter 12 & 13: 3D, vectors, vector functions

Chapter 14: Functions of more than 1 variable

Chapter 15: Integrating functions of more than 1 variable

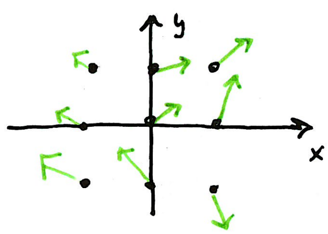
Chapter 16: Integrating vector Fields - vector functions of more than 1 variable

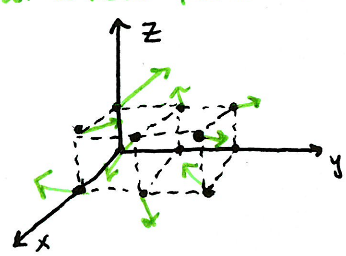
- Vector Fields: is a function whose domain is a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) who assigns to each point a vector in V_2 (or V_3).

★ See the Wind map on website ★ Page 1080 for examples

- Visualizing vector fields: Draw vectors at a few Points to visualize

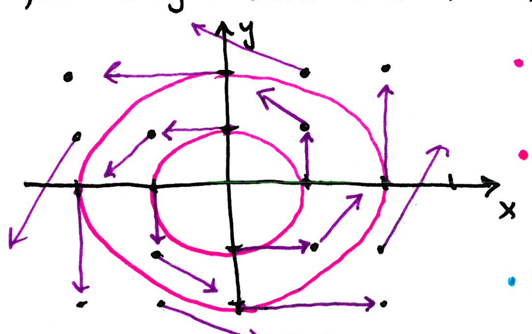
These dotted graphs are called lattice graphs





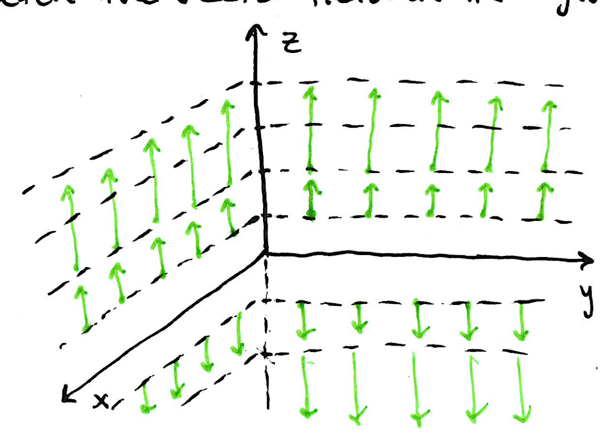
Example A vector field on $\mathbb{R}^2$ is defined by $\vec{F}(x,y) = \langle -y, x \rangle$. Describe $\vec{F}$ by sketching some of the vectors. What can be said about the magnitude of the vectors as you move away from the origin? What can you say about the flow/direction of the vectors?

x	y	$\vec{F}(x,y)$
0	1	$\langle -1, 0 \rangle$
1	0	$\langle 0, 1 \rangle$
-1	0	$\langle 0, -1 \rangle$
0	-1	$\langle 1, 0 \rangle$
1	1	$\langle -1, 1 \rangle$



- The magnitude increases as you move away from the origin.
- The vectors seem to flow in a counterclockwise direction about the origin.
- Each vector tangent to circle at (0,0)
  $\vec{x} \cdot \vec{F}(x,y) = 0$ ✓

Example Sketch the vector field on $\mathbb{R}^3$ given by $\vec{F}(x,y,z) = \langle 0, 0, z \rangle$.



Every plane $z=k$ has the same vector at every point $\langle 0, 0, k \rangle$

Section 16.1 - Vector Fields

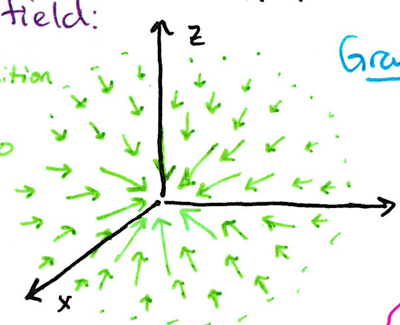
- Newton's Law of Gravitation:

Magnitude of the gravitational force between two objects with masses m and M is $|\vec{F}| = \frac{mMG}{r^2}$ where r is the distance between the objects and G is the gravitational constant.

Taking M to be located at the origin with $\vec{r} = \langle x, y, z \rangle$ the position vector for m then Gravitation force exerted on the second object acts towards the origin, that is in the $-\frac{\vec{r}}{|\vec{r}|}$ direction. Thus the gravitational force (field) is $\vec{F}(\vec{r}) = -\frac{mMG}{|\vec{r}|^2}\vec{r}$.

★ Give a rough sketch of this vector field:

- $-mMG\vec{r}$ means vectors point opposite of position
- Dividing by $|\vec{r}|^2$ means $|\vec{F}| \to \infty$ as $|\vec{r}| \to 0$



Gravitational Field

★ Demo 3D vector fields $1/r^2$ single

- Gradient Field:

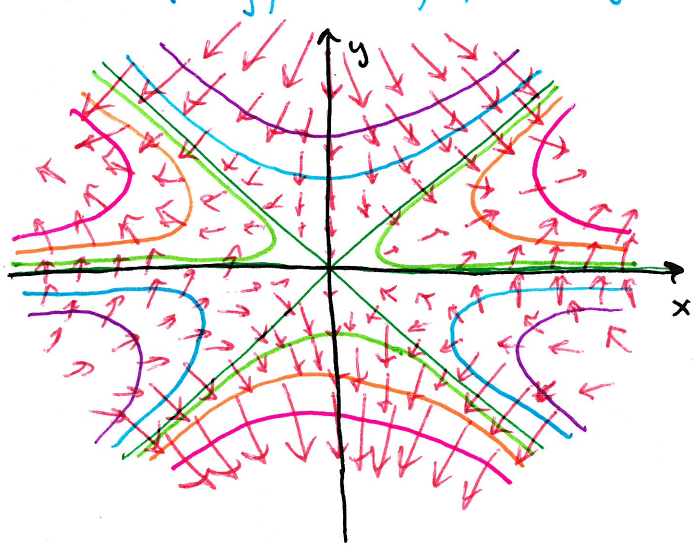
Very Important Vector field

Recall: $\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$ so the gradient of f is a vector field.

$\boxed{\text{Example}}$ Find the gradient field of $f(x,y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

★ Use 2D vector field plotter Scale = 0.01 Contours K = 100, 50, 1, 0, -50, -200

$$\nabla f = \langle 2xy,\ x^2 - 3y^2 \rangle = 2xy\,i + (x^2 - 3y^2)\,j$$



$f(x,y) = 100$

$f(x,y) = 50$

$f(x,y) = 1$

$f(x,y) = 0$

$f(x,y) = -50$

$f(x,y) = -200$

Section 16.1 - Vector fields

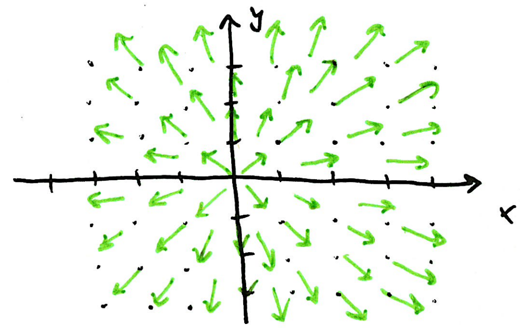
Example Find the gradient vector field of $f(x,y) = \sqrt{x^2+y^2}$ and sketch it.

$$\nabla f = \left\langle (x^2+y^2)^{-1/2}\cdot\tfrac{1}{2}(2x),\ (x^2+y^2)^{-1/2}\cdot\tfrac{1}{2}(2y)\right\rangle$$

$$= \left\langle x(x^2+y^2)^{-1/2},\ y(x^2+y^2)^{-1/2}\right\rangle$$

$$= (x^2+y^2)^{-1/2}\langle x,y\rangle$$

$$= \langle x,y\rangle / |\langle x,y\rangle| \quad \leftarrow \text{all unit vectors!}$$

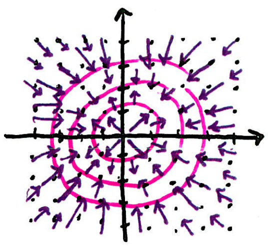


• Extra Examples

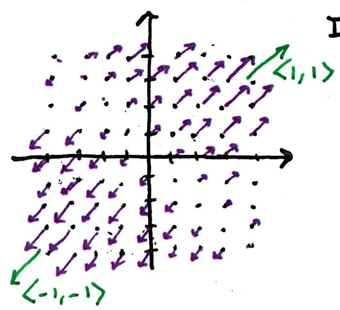
\# 29-32. Match the functions with the plots of their gradient vector fields I-IV

29. $f(x,y) = x^2+y^2$ 30. $f(x,y) = x(x+y)$ 31. $f(x,y) = (x+y)^2$ 32. $f(x,y) = \sin\sqrt{x^2+y^2}$

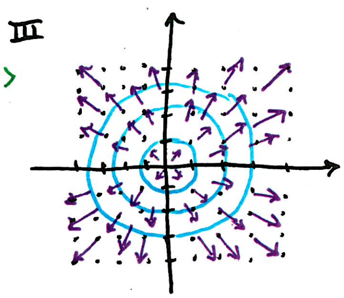
I



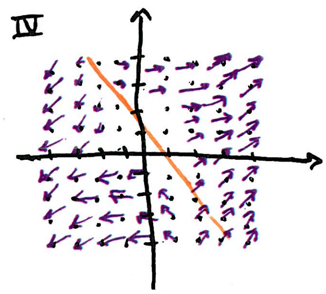
II



III



IV



29. level curves $k = x^2+y^2$ circles $\Rightarrow$ $\boxed{\text{III}}$

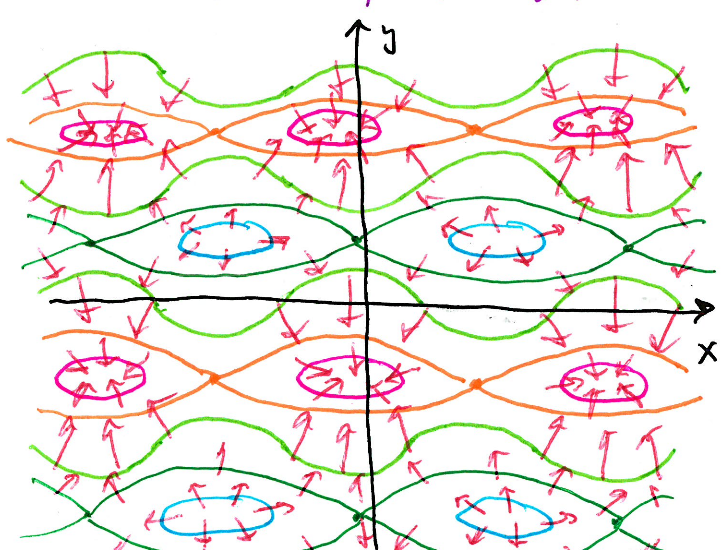
30. $\nabla f = \langle 2x+y,\ x\rangle$ Horizontal vectors when $y=-2x$ $\Rightarrow$ $\boxed{\text{IV}}$

31. $\nabla f = 2(x+y)\langle 1,1\rangle$ Only 2 directions $k\langle 1,1\rangle$ or $-k\langle 1,1\rangle \Rightarrow \boxed{\text{II}}$

32. level curves $k=\sin\sqrt{x^2+y^2}$ $\rightarrow c = x^2+y^2 \Rightarrow$ circles $\nabla f = \dfrac{\cos(|\langle x,y\rangle|)}{|\langle x,y\rangle|}\langle x,y\rangle$ $\boxed{\text{I}}$

\# 28. Plot the gradient vector field of f together with a contour map of f using the online plotters; $f(x,y) = \cos(x) - 2\sin(y)$.

$\nabla f = \langle -\sin(x),\ -2\cos(y)\rangle$ used scale $=0.5$ $k = 2, 1, 0, -1, -2$



$f(x,y)=2$

$f(x,y)=1$

$f(x,y)=0$

$f(x,y)=-1$

$f(x,y)=-2$