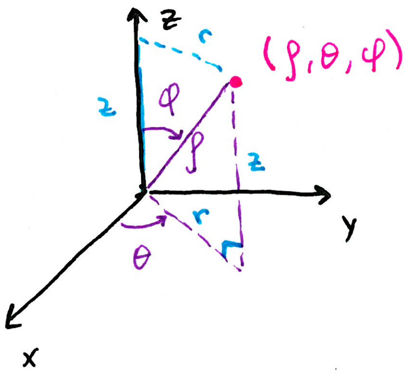
Section 15.9 - Spherical Coordinates

★ Useful for triple integrals over regions involving spheres or regions that are spherical.

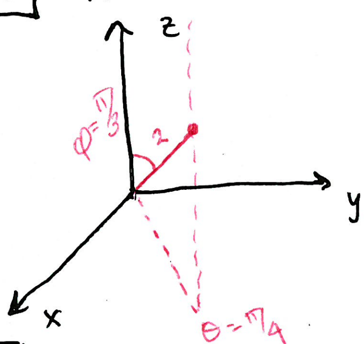


$x = r\cos\theta \quad y = r\sin\theta \quad z = \rho\cos\varphi$

$r = \rho\sin\varphi$

$$\boxed{\begin{array}{ccc} x = \rho\sin\varphi\cos\theta & y = \rho\sin\varphi\sin\theta & z = \rho\cos\varphi \\ \rho \geq 0 & \rho^2 = x^2+y^2+z^2 & 0 \leq \varphi \leq \pi \end{array}}$$

Example Plot $(2, \pi/4, \pi/3)$ and find the rectangular coordinates.



$x = 2\sin\pi/3\cos\pi/4 = 2\left(\frac{\sqrt{3}}{2}\right)\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{2}$

$y = 2\sin\pi/3\sin\pi/4 = \frac{\sqrt{6}}{2}$

$z = 2\cos\pi/3 = 1$

$\boxed{\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}, 1\right)}$

Example Convert $(0, 2\sqrt{3}, -2)$ to spherical coordinates.

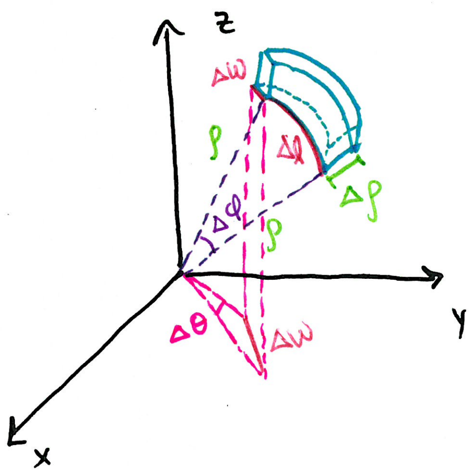
$\rho^2 = (0)^2 + (2\sqrt{3})^2 + (-2)^2 = 16 \Rightarrow \rho = 4$

$-2 = 4\cos\varphi \Rightarrow \varphi = 2\pi/3$

$0 = 4\sin(2\pi/3)\cos\theta \Rightarrow \theta = \pi/2$ or $\frac{3\pi}{2}$ in Q2 $\Rightarrow \theta = \pi/2$

$\boxed{(4, \pi/2, 2\pi/3)}$

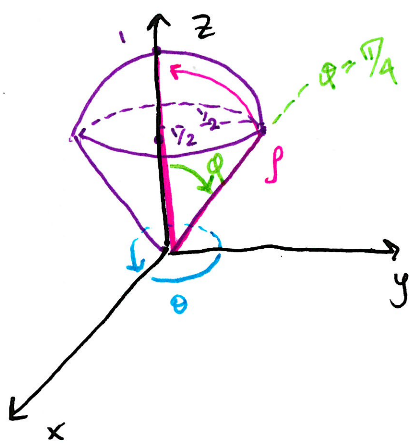
• Triple Integral in Spherical Coordinates:



$$\begin{aligned} \Delta V &= \Delta w\,\Delta l\,\Delta\rho \\ &= \frac{\Delta\theta}{2\pi}(2\pi r)\cdot\frac{\Delta\varphi}{2\pi}(2\pi\rho)\,\Delta\rho \\ &= r\rho\,\Delta\theta\,\Delta\varphi\,\Delta\rho \\ &= \rho^2\sin\varphi\,\Delta\rho\,\Delta\theta\,\Delta\varphi \end{aligned}$$

$$\boxed{E = [a,b]\times[\alpha,\beta]\times[c,d] \qquad \iiint_E f(x,y,z)\,dV = \int_c^d\int_\alpha^\beta\int_a^b f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi}$$

$\boxed{\text{Example}}$ Use spherical coordinates to find the volume of the solid above ① $z=\sqrt{x^2+y^2}$ and below ② $x^2+y^2+z^2=z$. $\left(x^2+y^2+(z-\frac{1}{2})^2=(\frac{1}{2})^2\right)$



$$E=\{(r,\theta,\varphi) \mid 0\le\theta\le 2\pi,\ 0\le\varphi\le\pi/4,\ 0\le\rho\le\cos\varphi\}$$

②: $\rho^2 = z = \rho\cos\varphi \Rightarrow \rho=\cos\varphi$

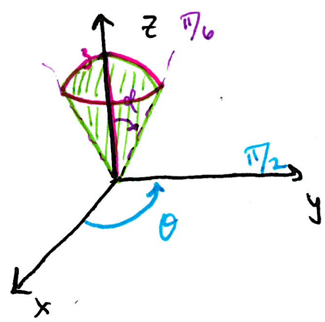
$$V=\iiint_E 1\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{\cos\varphi}\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$=\frac{2\pi}{3}\int_0^{\pi/4}\cos^3\varphi\sin\varphi\,d\varphi$$

$$=\frac{2\pi}{3}\cdot\frac{1}{4}\left(-\cos^4(\pi/4)+\cos^4(0)\right)=\boxed{\frac{\pi}{8}}$$

- Extra Examples

#17. Sketch the solid whose volume is given by $\int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$



$0\le\varphi\le\pi/6 \quad 0\le\theta\le\pi/2 \quad 0\le\rho\le 3$

#35. Find the volume and centroid of the solid E that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$.

$$V=\int_0^{\pi/4}\int_0^{2\pi}\int_0^1 \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi=\frac{2\pi}{3}\int_0^{\pi/4}\sin\varphi\,d\varphi=\boxed{\frac{2\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right)}$$

$$m=\rho\cdot V=\rho\,\frac{2\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right) \qquad M_{xy}=2\pi\int_0^{\pi/4}\int_0^1\rho^3\sin\varphi\cos\varphi\,d\rho\,d\varphi$$

$$=\frac{\pi}{2}\cdot\frac{1}{2}\left(\frac{1}{2}\right)=\frac{\pi}{8}$$

$$M_{yz}=\int_0^{\pi/4}\int_0^{2\pi}\int_0^1\rho^3\sin^2\varphi\cos\theta\,d\rho\,d\theta\,d\varphi=0$$

$$M_{xz}=\int_0^{\pi/4}\int_0^{2\pi}\int_0^1\rho^3\sin^2\varphi\sin\theta\,d\rho\,d\theta\,d\varphi=0$$

$\boxed{\text{Centroid: } \left(0,0,\ 3\rho/8(2-\sqrt{2})\right)}$