# Section 15.7 – Triple Integrals



★ We used double integrals to compute the mass, center of mass and moments of laminas (thin plates) but what about solid objects in 3D?

Will need to add in 3 directions → Triple Integral

- Triple Integrals over a rectangular box:

$$B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ e \le z \le f\}$$

$$\Delta V = \Delta x \Delta y \Delta z$$

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\,\Delta V$$

- Fubini's Theorem for Triple Integrals: for $f(x,y,z)$ continuous on $B = [a,b] \times [c,d] \times [e,f]$

$$\iiint_B f(x,y,z)\,dV = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dx\,dy\,dz$$

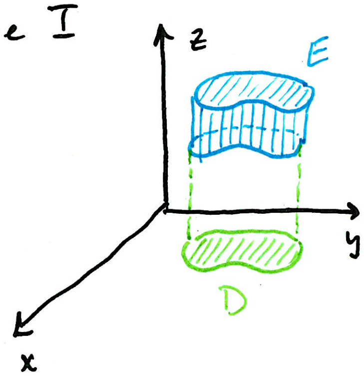
★ Order can be changed

How many way?

$3 \cdot 2 \cdot 1 = 3! = 6$

- Triple Integrals over a 3D region: $\iiint_E f(x,y,z)\,dV =$

Type I

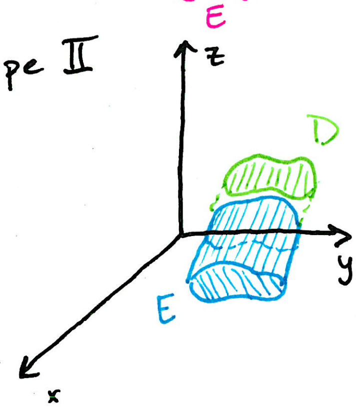



$g(x,y) \le z \le h(x,y)$

D in xy-plane

$$\iint_D \left[ \int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz \right] dA$$

Type II

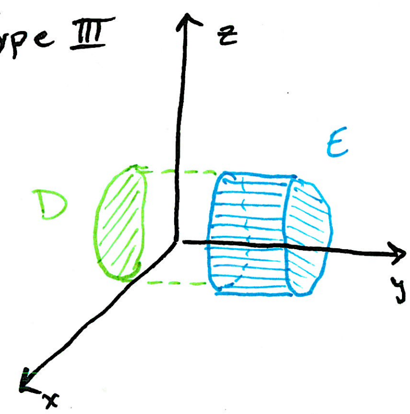



$g(y,z) \le x \le h(y,z)$

D in yz-plane

$$\iint_D \left[ \int_{g(y,z)}^{h(y,z)} f(x,y,z)\,dx \right] dA$$

Type III




$g(x,z) \le y \le h(x,z)$

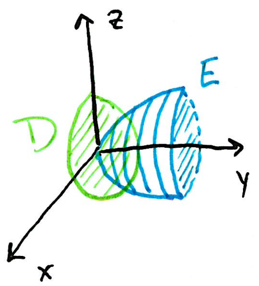
D in xz-plane

$$\iint_D \left[ \int_{g(x,z)}^{h(x,z)} f(x,y,z)\,dy \right] dA$$

# Section 15.7 - Triple Integrals



**Example** Evaluate $\iiint_E \sqrt{x^2+z^2}\, dV$, where $E$ is bounded by the paraboloid $y = x^2+z^2$ and the plane $z=4$

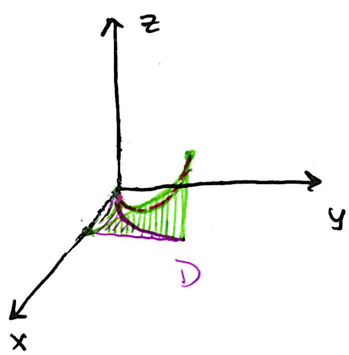



$x^2+z^2 \le y \le 4$ $\qquad D = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 2\}$

$$\iint_D \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\, dA = \int_0^{2\pi}\int_0^{2} (4-(x^2+z^2))\sqrt{x^2+z^2}\, r\, dr\, d\theta$$

$$= 2\pi \int_0^2 4r^2 - r^4\, dr = \boxed{2\pi\left(\frac{4}{3}(2)^3 - \frac{(2)^5}{5}\right)}$$

**Example** Express the integral ① $\int_0^1 \int_0^{x^2} \int_0^{y} f(x,y,z)\, dz\, dy\, dx$ as a triple Integral the 5 other ways.



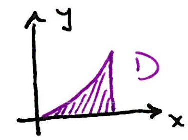

$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x^2\}$

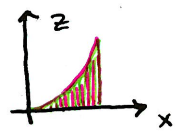
$0 \le z \le y$

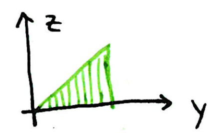
② $\int_0^1 \int_{\sqrt{y}}^1 \int_0^{y} f(x,y,z)\, dz\, dx\, dy =$ ③ $\int_0^1 \int_z^1 \int_{\sqrt{z}}^1 f(x,y,z)\, dx\, dy\, dz$

$=$ ④ $\int_0^1 \int_0^{y} \int_{\sqrt{z}}^1 f(x,y,z)\, dx\, dz\, dy =$ ⑤ $\int_0^1 \int_{\sqrt{z}}^1 \int_0^{x^2} f(x,y,z)\, dy\, dx\, dz$

$=$ ⑥ $\int_0^1 \int_0^{x^2} \int_z^1 f(x,y,z)\, dy\, dz\, dx$




- Applications of Triple Integrals:

$E$ a closed region in 3D then: $V(E) = \iiint_E 1\, dV$

Mass of $E$ with density function $\rho(x,y,z)$: $m = \iiint_E \rho(x,y,z)\, dV$

Moments about the Coordinate planes:

$$M_{xy} = \iiint_E z\,\rho(x,y,z)\, dV \qquad M_{xz} = \iiint_E y\,\rho(x,y,z)\, dV \qquad M_{yz} = \iiint_E x\,\rho(x,y,z)\, dV$$

Center of mass: $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right)$

Moments of Inertia about the Coordinate axes:

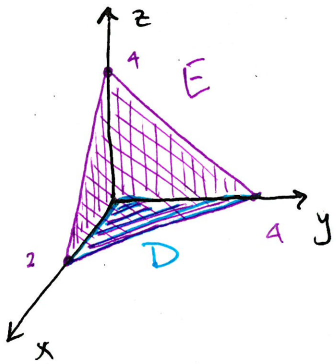
$$I_x = \iiint_E (y^2+z^2)\,\rho(x,y,z)\, dV \qquad I_y = \iiint_E (x^2+z^2)\,\rho(x,y,z)\, dV \qquad I_z = \iiint_E (x^2+y^2)\,\rho(x,y,z)\, dV$$

# Section 15.7 - Triple Integrals



- Extra Examples:

\# 19. Find the volume of the region bounded by $2x+y+z=4$ and the coordinate planes as a triple integral.



$$D = \{(x,y) \mid 0 \le x \le 2 \quad 0 \le y \le -2x+4\}$$

$$0 \le z \le 4-2x-y$$

$$V = \iiint_E 1\, dV = \int_0^2 \int_0^{-2x+4} \int_0^{4-2x-y} 1\, dz\, dy\, dx = \int_0^2 \int_0^{-2x+4} (4-2x-y)\, dy\, dx$$

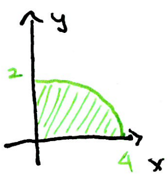
$$= \int_0^2 4y - 2xy - \frac{y^2}{2} \Big|_0^{4-2x} dx = \int_0^2 16 - 16x + 4x^2 - (2-x)^2\, dx$$

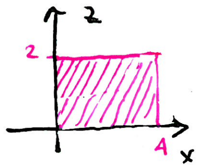
$$= 16 - \frac{8}{3} = \boxed{16/3}$$

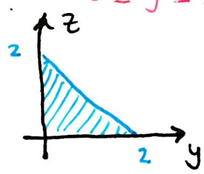
\# 28. Sketch the solid whose volume is given by $\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy$.

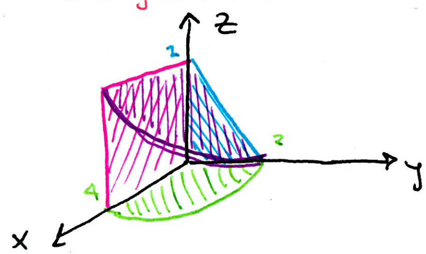
$0 \le y \le 2$ $\quad$ $0 \le z \le 2-y$ $\quad$ $0 \le x \le 4-y^2$

$0 \le y \le 2-z$ $\quad$ $0 \le y \le \sqrt{4-x}$



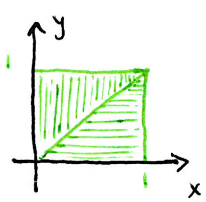


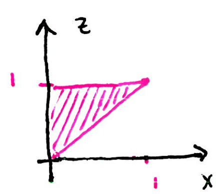


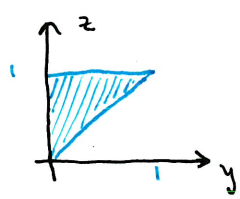


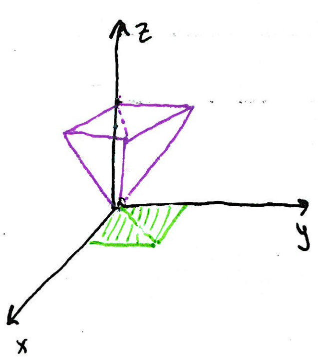
\#36. Write the 5 other iterated integrals for: ① $\int_0^1 \int_y^1 \int_0^z f(x,y,z)\, dx\, dz\, dy$

$0 \le y \le 1$ $\quad$ $y \le z \le 1$ $\quad$ $0 \le x \le z$ $\quad$ $z = x$ and $z = y$









② $\int_0^1 \int_0^z \int_0^z f\, dx\, dy\, dz$ = ③ $\int_0^1 \int_x^1 \int_0^z f\, dy\, dz\, dx$ = ④ $\int_0^1 \int_0^z \int_0^z f\, dy\, dx\, dz$

⑤ $\int_0^1 \int_y^1 \int_x^1 f\, dz\, dx\, dy + \int_0^1 \int_0^y \int_y^1 f\, dz\, dx\, dy$ = ⑥ $\int_0^1 \int_0^x \int_x^1 f\, dz\, dy\, dx + \int_0^1 \int_x^1 \int_y^1 f\, dz\, dy\, dx$