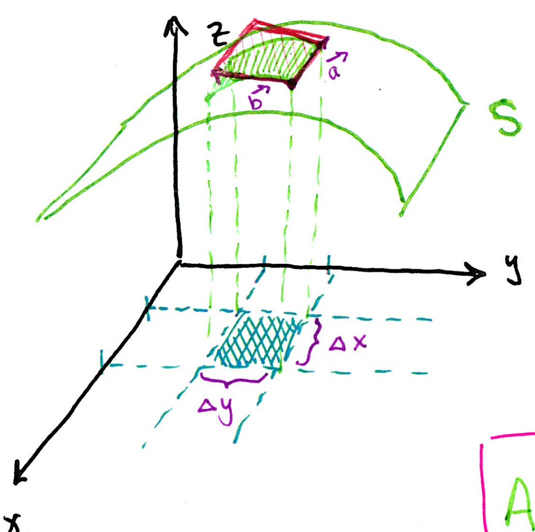
# Section 15.6 - Surface Area

MVC

Consider the surface $S$ given by a continuous function $z=f(x,y)$ with partial derivatives.

**Goal:** Find the Surface Area of $S$ over a region $D$.

**Idea:**
1. Split $D$ into small rectangles
2. Compute area of tangent plane to surface over small rectangle
3. Sum all areas
4. As number of small rectangles $\to\infty$ get the Surface Area




$\vec{a} = \langle \Delta x, 0, f_x(x,y)\Delta x\rangle$

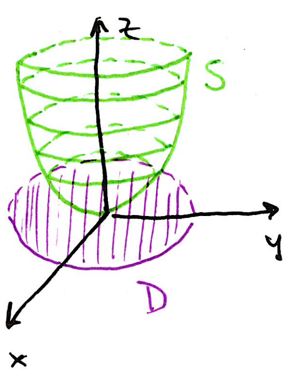
$\vec{b} = \langle 0, \Delta y, f_y(x,y)\Delta y\rangle$

Area of tangent Rectangle:

$$|\vec{a}\times\vec{b}| = |\langle -f_x(x,y)\Delta x\Delta y, -f_y(x,y)\Delta x\Delta y, \Delta x\Delta y\rangle|$$
$$= \sqrt{f_x^2(x,y)+f_y^2(x,y)+1}\,\Delta A$$

$$\boxed{A(S) = \iint_D \sqrt{f_x^2(x,y)+f_y^2(x,y)+1}\,dA}$$

**Example** Find the area of part of the paraboloid $z=x^2+y^2$ that lies under the plane $z=4$.




$D = \{(r,\theta) \mid 0\le\theta\le 2\pi,\ 0\le r\le 2\}$

$$A(S) = \int_0^{2\pi}\int_0^2 \sqrt{1+(2x)^2+(2y)^2}\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\,r\,dr\,d\theta$$
$$= 2\pi \cdot \frac{\frac{2}{3}(1+4r^2)^{3/2}}{8}\Bigg|_0^2 = \boxed{\frac{\pi}{6}\left(17^{3/2}-1\right)}$$

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- Extra Examples

#9. The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$. Find the area.

$$D = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1\}$$

$$A(S) = \int_0^{2\pi}\int_0^1 \sqrt{1+(y)^2+(x)^2}\, r\,dr\,d\theta = 2\pi \int_0^1 \sqrt{1+r^2}\, r\,dr\,d\theta$$

$$= 2\pi \left.\frac{(1+r^2)^{3/2}}{3}\right|_0^1 = \boxed{\frac{2\pi}{3}\left(2^{3/2}-1\right)}$$

#21. Show that the area of the part of the plane $z = ax + by + c$ that projects onto a region $D$ in the xy-plane with area $A(D)$ is $\sqrt{a^2+b^2+1} \cdot A(D)$.

$$A(S) = \iint_D \sqrt{1+(a)^2+(b)^2}\, dA$$

$$= \sqrt{1+a^2+b^2} \iint_D 1\, dA$$

$$= \sqrt{1+a^2+b^2}\, A(D) \quad \blacksquare$$

#24. Find the area of the surface created when $y^2 + z^2 = 1$ intersects $x^2 + z^2 = 1$.

By Symmetry the surface Area = 4 x Surface Area that intersects the positive y-axis

Projecting this face into the xz-plane gives $D$: $x^2 + z^2 \le 1$ $\quad y = \sqrt{1-z^2}$

$$A(S) = 4\iint_{x^2+z^2\le 1} \sqrt{1+\left(\frac{-z}{(1-z^2)^{1/2}}\right)^2}\, dA = 4\iint_{x^2+z^2\le 1} \frac{1}{\sqrt{1-z^2}}\, dA = 4 \cdot 4 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\, dx\,dz$$

* Note improper integral at $z = 1$:

$$A(S) = 16 \cdot \lim_{t\to 1^-} \int_0^t \int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\, dx\,dz = 16 \cdot \lim_{t\to 1^-} \int_0^t 1\, dx = \boxed{16}$$