Section 15.5 - Applications of Double Integrals

Physical Applications: Computing Mass, electric charge, Center of mass, inertia

Already Seen: Average value of a function $f_{ave} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA$

Consider a thin plate (called a lamina) with variable density occupying a region D in the xy-plane:

- **Goals**: Given a density function $\rho(x,y)$ for a Lamina find:
  1. total mass of Lamina
  2. Moments of the Lamina
  3. Center of mass
  4. Moments of Inertia

- **Total mass of lamina**: Sum masses over ΔA

  In general density = $\frac{\text{mass}}{\text{volume}}$ but for thin plate density = $\frac{\text{mass}}{\text{area}}$

  $\Delta m = \rho(x,y)\Delta A$ $$\boxed{m = \iint_D \rho(x,y)\,dA}$$

- **Moments of Lamina**: product of mass and its directed distance from an axis.
  (measures the tendency for the plate to rotate about the axis)

  About x-axis: $\Delta m_x = \Delta m \cdot y = y \cdot \rho(x,y)\Delta A$

  $$\boxed{M_x = \iint_D y\,\rho(x,y)\,dA}$$

  Tendency to rotate about x-axis

  About y-axis: $\Delta m_y = \Delta m \cdot x = x \cdot \rho(x,y)\Delta A$

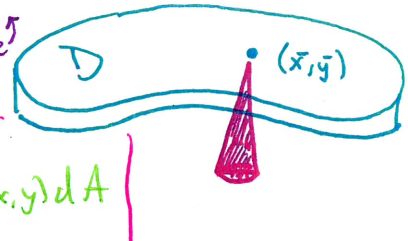
  $$\boxed{M_y = \iint_D x\,\rho(x,y)\,dA}$$

  Tendency to rotate about y-axis

- **Center of mass**: point where plate balances horizontally

  ★ Point may not be on plate Ex. Horseshoe

  point $(\bar{x},\bar{y})$ such that $m\bar{x} = M_y$, $m\bar{y} = M_x$



$$\boxed{\bar{x} = \frac{M_y}{M} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA} \qquad \boxed{\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA}$$

★ See moments demo on website / Paper plate demo

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MVC

- **Moments of Inertia:**

  ★ mass determines force needed for an acceleration

  ★ So Inertia determines torque needed for an angular acceleration

About x-axis: $\Delta I_x = \Delta m \cdot y^2 = y^2 \rho(x,y)\,\Delta A$

$$\boxed{I_x = \iint_D y^2 \rho(x,y)\,dA}$$

About y-axis: $\Delta I_y = \Delta m \cdot x^2 = x^2 \rho(x,y)\,\Delta A$

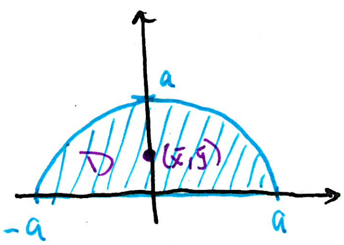
$$\boxed{I_y = \iint_D x^2 \rho(x,y)\,dA}$$

About origin: $\Delta I_0 = \Delta m\,(x^2+y^2) = (x^2+y^2)\rho(x,y)\,\Delta A$

$$\boxed{I_0 = \iint_D (x^2+y^2)\rho(x,y)\,dA}$$

Note: $I_0 = I_x + I_y$

Example The Density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass.



$D = \{(r,\theta) \mid 0 \le r \le a,\ 0 \le \theta \le \pi\}$ $\rho(x,y) = K\sqrt{x^2+y^2}$, K constant of proportionality

$\rho(r,\theta) = Kr$

$$M = \iint_D r\,dA = \int_0^{\pi}\int_0^{a} Kr^2\,dr\,d\theta = \frac{K\pi a^3}{3}$$

By symmetry know $\bar{x} = 0$

$$\bar{y} = \frac{3}{K\pi a^3}\int_0^{\pi}\int_0^{a} Kr^2 (r\sin\theta)\,dr\,d\theta = \frac{3}{K\pi a^3}\left(\frac{Ka^4}{4}\right)(2) = \frac{3a}{2\pi}$$

Center of mass: $(\bar{x},\bar{y}) = \boxed{\left(0, \frac{3a}{2\pi}\right)}$

Example Find the moment of inertia I_0 of a homogeneous D with density ρ Center $(0,0)$ and radius a. Use I_0 to determine I_x, I_y.

$D = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le a\}$

$$I_0 = \int_0^{2\pi}\int_0^{a} r^2 \rho\, r\,dr\,d\theta = \boxed{2\pi\rho\frac{a^4}{4}}$$

By Symmetry of D: $I_x = I_y$ and as $I_0 = I_x + I_y$

$$I_x = I_y = \boxed{\pi\rho\frac{a^4}{4}}$$

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- Extra Examples

\# 11. A lamina occupies the first quadrant of $x^2+y^2 \le 1$. Find its center of mass if the density at any point is proportional to its distance from the x-axis.

$D = \{(r,\theta) \mid 0 \le \theta \le \pi/2,\ 0 \le r \le 1\}$ $\quad \rho(x,y) = ky \quad \rho(r,\theta) = kr\sin\theta$

$$M = \int_0^{\pi/2}\int_0^1 Kr^2\sin\theta \, dr\, d\theta = \frac{K}{3}$$

$$\bar{x} = \frac{3}{K}\int_0^{\pi/2}\int_0^1 r\cos\theta \cdot Kr\sin\theta \cdot r\, dr\, d\theta = \frac{3}{K}\cdot\frac{K}{4}\cdot\frac{1}{2} = \frac{3}{8}$$

$$\bar{y} = \frac{3}{K}\int_0^{\pi/2}\int_0^1 r\sin\theta \cdot Kr\sin\theta \cdot r\, dr\, d\theta = \frac{3K}{4K}\int_0^{\pi/2} \frac{1-\cos(2\theta)}{2}\, d\theta$$

$$= \frac{3}{8}\left(\theta - \cancel{\frac{\sin(2\theta)}{2}}^{\,0}\right)\Bigg|_0^{\pi/2} = \frac{3\pi}{16} \qquad \boxed{(\bar{x},\bar{y}) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)}$$

\# 12. Lamina from # 11 but the density at any point is proportional to the square of its distance from the origin.

$$\rho(x,y) = (x^2+y^2)K \quad \rho(r,\theta) = (r^2)K$$

$$M = \int_0^{\pi/2}\int_0^1 Kr^3\, dr\, d\theta = \frac{K}{4}$$

$$\bar{x} = \frac{4}{K}\int_0^{\pi/2}\int_0^1 r\cos\theta \cdot Kr^3\, dr\, d\theta = \frac{4}{5}$$

$$\bar{y} = \frac{4}{K}\int_0^{\pi/2}\int_0^1 r\sin\theta \cdot Kr^3\, dr\, d\theta = \frac{4}{5}$$

$$\boxed{(\bar{x},\bar{y}) = \left(\frac{4}{5}, \frac{4}{5}\right)}$$

\# 18. Find I_0, I_x, I_y for the lamina in # 12.

$$I_x = \int_0^{\pi/2}\int_0^1 r^2\cos^2\theta\, r^2 \cdot K \cdot r\, dr\, d\theta = \frac{K}{6}\int_0^{\pi/2} \frac{1+\cos(2\theta)}{2}\, d\theta$$

$$= \frac{K}{6}\cdot\frac{1}{2}\left(\theta + \cancel{\frac{\sin(2\theta)}{2}}^{\,0}\right)\Bigg|_0^{\pi/2} = \boxed{\frac{K}{24}}$$

$$I_y = \int_0^{\pi/2}\int_0^1 r^2\sin^2\theta\, r^2 \cdot Kr\, dr\, d\theta = \frac{K}{6}\int_0^{\pi/2} \frac{1-\cos(2\theta)}{2} = \boxed{\frac{K}{24}}$$

$$I_0 = I_x + I_y = \boxed{\frac{K}{12}}$$