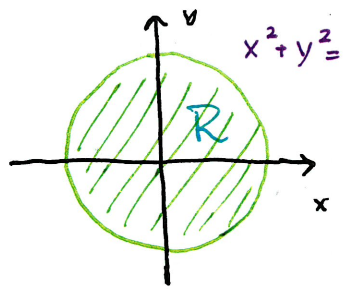
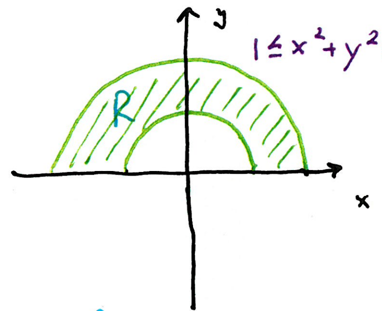
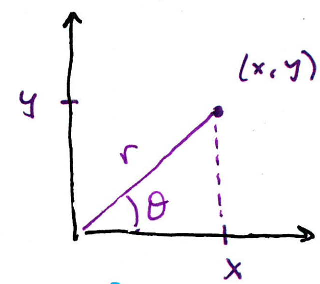
Section 15.4 - Double Integrals in Polar Coordinates

★ Regions that are circular in nature are difficult to describe in Cartesian Coordinates but are easy in Polar Coordinates.





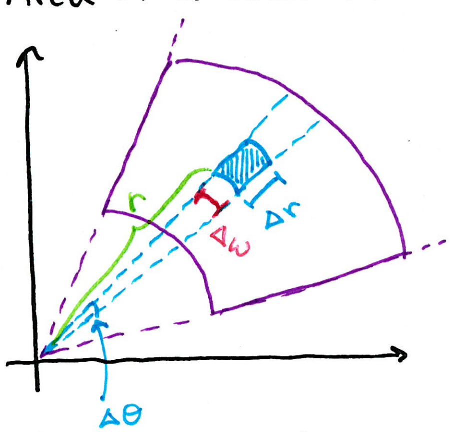


Rectangular: $R = \{(x,y) \mid |x| \le 1, |y| \le \sqrt{1-x^2}\}$ $R = \{(x,y) \mid 1 \le |x| \le 2, \sqrt{1-x^2} \le y \le \sqrt{4-x^2}\}$

Polar: $R = \{(r,\theta) \mid |r| \le 1, 0 \le \theta \le 2\pi\}$ $R = \{(r,\theta) \mid 1 \le |r| \le 2, 0 \le \theta \le \pi\}$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = r^2$

- Area of a small Polar Rectangle:



$\Delta A = \Delta x \Delta y$

$\Delta A = \Delta\omega \Delta r$

$\Delta A = 2\pi r \dfrac{\Delta\theta}{2\pi} \Delta r$

$\Delta A = r \Delta r \Delta\theta$

Important Identities:

1. $\cos^2\theta + \sin^2\theta = 1$
2. $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$
3. $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$

- Change to Polar Coords in a Double Integral:

★ Watch demo on Website

f continuous on $R = \{(r,\theta) \mid 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$ with $0 \le \beta - \alpha \le 2\pi$ then

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Example Evaluate $\iint_R (3x+4y^2)\,dA$ where R is the region in the upper-half plane bounded by $x^2+y^2=1$ and $x^2+y^2=4$.

$$\iint_R (3x+4y^2)\,dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\,dr\,d\theta = \int_0^\pi (7\cos\theta + 15\sin^2\theta)\,d\theta$$

$$= \cancel{7\sin\theta}\Big|_0^\pi{}^{0} + 15\int_0^\pi \frac{1-\cos(2\theta)}{2}\,d\theta$$

$$= \frac{15}{2}\left(\theta - \frac{\cancel{\sin(2\theta)}^{0}}{2}\right)\Bigg|_0^\pi = \boxed{\frac{15\pi}{2}}$$

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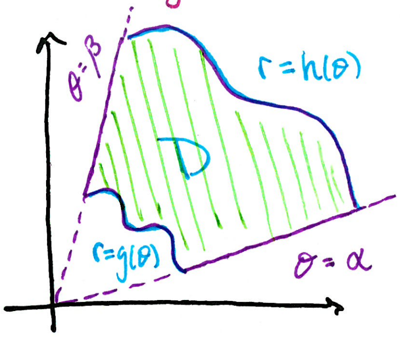
★ Polar regions but not polar rectangles:



f continuous on $D=\{(r,\theta)\mid \alpha\le\theta\le\beta,\ g(\theta)\le r\le h(\theta)\}$

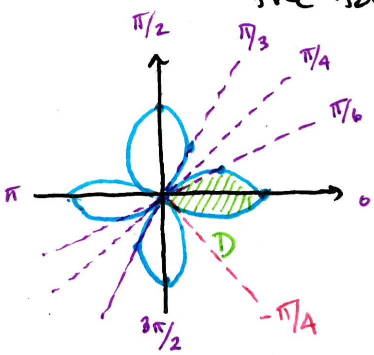
then $\boxed{\displaystyle\iint_D f(x,y)\,dA=\int_\alpha^\beta\int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$

Area of D: when $f(x,y)=1$

$$A(D)=\int_\alpha^\beta\int_{g(\theta)}^{h(\theta)} r\,dr\,d\theta=\int_\alpha^\beta \tfrac{1}{2}(h(\theta))^2-\tfrac{1}{2}(g(\theta))^2\,d\theta$$

Area between polar curves from Calculus

Example Use a double integral to find the area enclosed by one loop of the four-leaf rose curve: $r=\cos(2\theta)$.



$$A(D)=\int_{-\pi/4}^{\pi/4}\int_0^{\cos(2\theta)} r\,dr\,d\theta=\frac{1}{2}\int_{-\pi/4}^{\pi/4}\cos^2(2\theta)\,d\theta$$

$$=\frac{1}{2}\cdot\frac{1}{2}\int_{-\pi/4}^{\pi/4} 1+\cos(4\theta)\,d\theta$$

$$=\frac{1}{4}\left(\theta+\cancel{\frac{\sin(4\theta)}{4}}^{\,0}\right)\Big|_{-\pi/4}^{\pi/4}=\boxed{\frac{\pi}{8}}$$

Example Find the volume of the solid that lies under $z=x^2+y^2$ above the xy-plane, and inside the cylinder $x^2+y^2=2x$.

$$\iint_R x^2+y^2\,dA=\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}(r^2)\,r\,dr\,d\theta$$

$\boxed{\text{Bounds}}$

$r^2\cos^2\theta+r^2\sin^2\theta=2r\cos\theta$

$r^2=2r\cos\theta$

$r=0$ or $r=2\cos\theta$

$0=2\cos\theta$

$\theta=-\pi/2,\ \pi/2$

$$=4\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta=\int_{-\pi/2}^{\pi/2}(1+\cos(2\theta))^2\,d\theta$$

$$=\int_{-\pi/2}^{\pi/2}(1+2\cos(2\theta)+\cos^2(2\theta))\,d\theta$$

$$=\pi+\cancel{\frac{2\sin(2\theta)}{2}}^{\,0}\Big|_{-\pi/2}^{\pi/2}+\int_{-\pi/2}^{\pi/2}\frac{1+\cos(4\theta)}{2}\,d\theta$$

$$=\pi+\frac{1}{2}\left(\theta+\cancel{\frac{\sin(4\theta)}{4}}^{\,0}\right)\Big|_{-\pi/2}^{\pi/2}=\boxed{\frac{3\pi}{2}}$$

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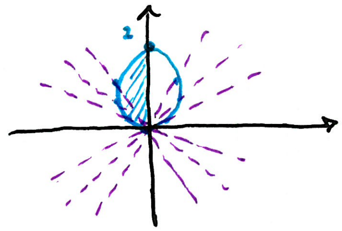
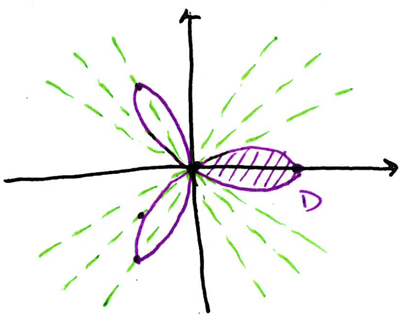
- Extra Examples

#6. Sketch the region whose area is given by $\int_{\pi/2}^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta$.

$\pi/2 \le \theta \le \pi \qquad 0 \le r \le 2\sin\theta$



#15. Find the area of one loop of $r=\cos(3\theta)$ using a double integral.



Bounds $0=\cos(3\theta)$

$3\theta = \pm\pi/2$

$\theta = \pm\pi/6$

Symmetry

$$A(D) = \int_{-\pi/6}^{\pi/6}\int_0^{\cos(3\theta)} r\,dr\,d\theta = \int_0^{\pi/6} \cos^2(3\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6} 1+\cos(6\theta)\,d\theta = \frac{1}{2}\left(\theta + \frac{\sin(\cancel{6}\theta)}{\cancel{6}}\right)\Big|_0^{\pi/6} = \boxed{\frac{\pi}{12}}$$

#25. Find the volume of the solid above the cone $z=\sqrt{x^2+y^2}$ and below $x^2+y^2+z^2=1$.

$z \le \sqrt{1-x^2-y^2} \qquad z \ge \sqrt{x^2+y^2} \qquad$ Bounds: $r^2+r^2=1 \Rightarrow 0 \le r \le \sqrt{2}/2$

$z \le \sqrt{1-r^2} \qquad z \ge r \qquad 0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^{\sqrt{2}/2}\left(\sqrt{1-r^2}-r\right) r\,dr\,d\theta = 2\pi\int_0^{\sqrt{2}/2} r(1-r^2)^{1/2} - r^2\,dr$$

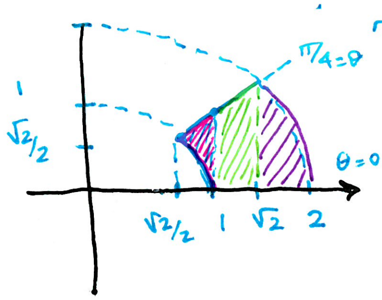
$$= 2\pi\left[\frac{\frac{2}{3}(1-r^2)^{3/2}}{(-2)} - \frac{r^3}{3}\right]\Bigg|_0^{\sqrt{2}/2} = 2\pi\left(-\frac{1}{3}\left(\frac{1}{2}\right)^{3/2} - \frac{1}{3}\left(\frac{1}{2}\right)^{3/2} + \frac{1}{3}\right) = \boxed{\frac{2\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right)}$$

#39. Use polar coords to combine the sum into one double integral.

$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_1^{\sqrt{2}}\int_0^{x} xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} xy\,dy\,dx$$

$\sqrt{1-x^2} \le y \le x \qquad 1 \le x \le \sqrt{2} \quad 0 \le y \le x \qquad \sqrt{2} \le x \le 2 \quad 0 \le y \le \sqrt{4-x^2}$

$1/\sqrt{2} \le x \le 1$



$$= \boxed{\int_0^{\pi/4}\int_1^2 r\cos\theta\, r\sin\theta\, r\,dr\,d\theta}$$