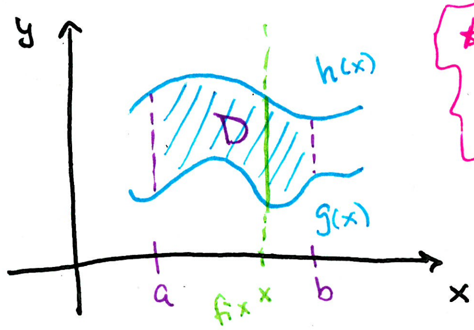
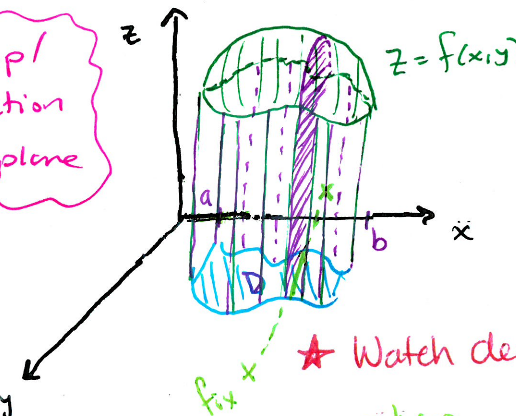
Section 15.3 - Double Integrals Over General Regions

★ Want to integrate over regions of a general shape

- **Type I** - Regions $D = \{(x,y) \mid a \le x \le b,\ g(x) \le y \le h(x)\}$



★ Have a top/bottom function in the xy plane

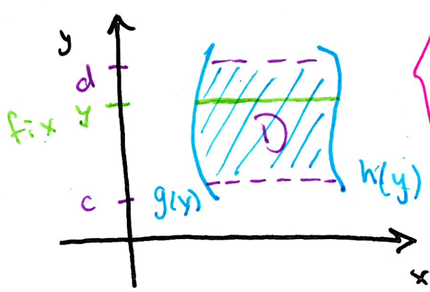


★ Watch demo on website

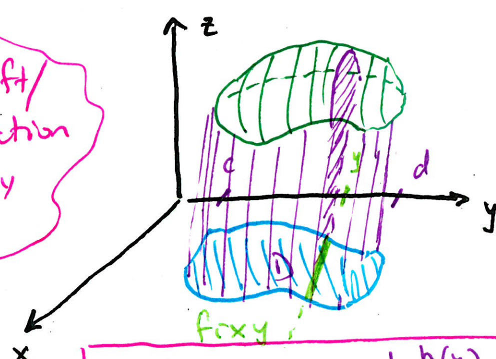
Fix x, find area of slice of $z = f(x,y)$: $A(x) = \int_{g(x)}^{h(x)} f(x,y)\,dy$

Suming these areas as x varries on $[a,b]$: $\boxed{\iint_D f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx}$

- **Type II** - Regions $D = \{(x,y) \mid c \le y \le d,\ g(y) \le x \le h(y)\}$

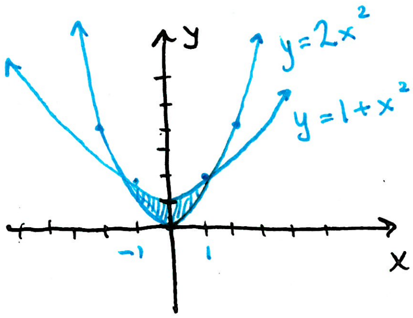


★ Have a left/right function in the xy plane



$A(y) = \int_{g(y)}^{h(y)} f(x,y)\,dx \rightarrow \boxed{\iint_D f(x,y)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy}$

Example Evaluate $\iint_D (x+2y)\,dA$ where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.



$D = \{(x,y) \mid -1 \le x \le 1,\ 2x^2 \le y \le 1 + x^2\}$ Type I

$$\iint_D (x+2y)\,dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx = \int_{-1}^{1} (x + y^2)\Big|_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^{1} x + 1 + 2x^2 + x^4 - 4x^4\,dx = \boxed{\frac{32}{15}}$$

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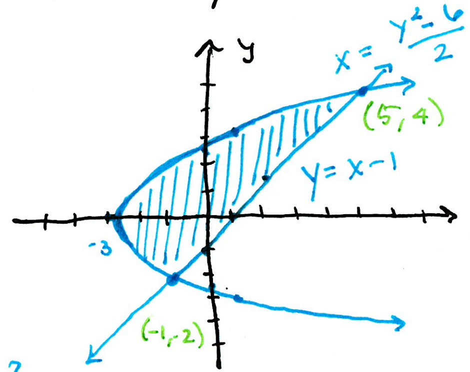
Example Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y=x-1$ and the parabola $y^2=2x+6$.



Intersection points: $(x-1)^2 = 2x+6$

$$x^2-2x+1 = 2x+6$$

$$x^2-4x-5=0$$

$$x=-1,\,5$$

Type II $D=\left\{(x,y) \mid -2 \le y \le 4,\ \frac{y^2-6}{2} \le x \le y+1\right\}$

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{y^2-6}{2}}^{y+1} xy\,dx\,dy = \int_{-2}^{4} \frac{y}{2}(y+1)^2 - \frac{y}{2}\left(\frac{y^2-6}{2}\right)^2 dy$$

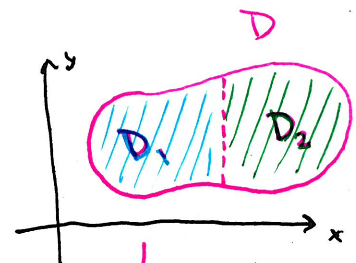
$$=\frac{1}{2}\int_{-2}^{4} y^3+2y^2+y \;-\; \frac{1}{8}\frac{(y^2-6)^3}{3\cdot 2}\Big|_{-2}^{4} = \frac{1}{2}\left[\frac{y^4}{4}+\frac{2y^3}{3}+\frac{y^2}{2}\right]\Big|_{-2}^{4} - \frac{125}{6} + \left(-\frac{1}{6}\right) = \boxed{36}$$

- **Properties of Double Integrals:**

① $\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$

② $\iint_D c\cdot f(x,y)\,dA = c\cdot\iint_D f(x,y)\,dA$

③ $f(x,y) \ge g(x,y)$ on D, then $\iint_D f(x,y)\,dA \ge \iint_D g(x,y)\,dA$



④ $\iint_D 1\,dA = \text{Area}(D)$

⑤ If $D=D_1\cup D_2$ and $D_1\cap D_2$ only on their boundary then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

⑥ If $m \le f(x,y) \le M$ on D then $m\cdot\text{Area}(D) \le \iint_D f(x,y)\,dA \le M\cdot\text{Area}(D)$

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- Extra Examples

#17. $\iint_D x\cos y \, dA$, D is bounded by $y=0$, $y=x^2$, $x=1$

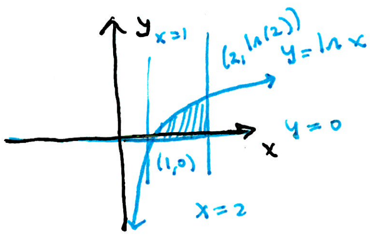
$$= \int_0^1 \int_0^{x^2} x\cos(y)\,dy\,dx = \int_0^1 x\sin(x^2)\,dx = -\frac{\cos(x^2)}{2}\Big|_0^1 = \boxed{-\frac{\cos(1)}{2} + \frac{1}{2}}$$

21. $\iint_D (2x-y)\,dA$, D bonded by circle at $(0,0)$ with radius 2.

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x-y)\,dy\,dx = \int_{-2}^{2} 2x(2\sqrt{4-x^2})\,dx$$

$$= -4 \cdot \frac{2}{3}\cancel{\frac{(4-x^2)^{3/2}}{2}}^{\,0}\Big|_{-2}^{2} = \boxed{0}$$

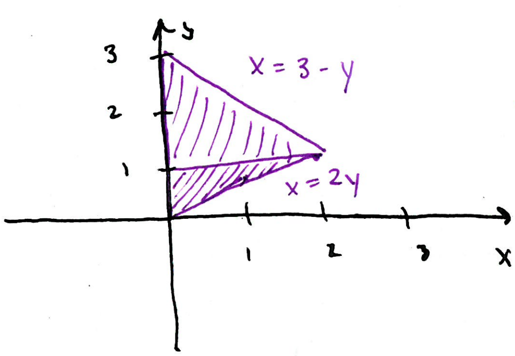
47. Sketch the region and reverse the order of integration $\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx$.



$$\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx = \boxed{\int_0^{\ln(2)} \int_{e^x}^{2} f(x,y)\,dx\,dy}$$

62. $\iint_D f(x,y)\,dA = \int_0^1 \int_0^{2y} f(x,y)\,dx\,dy + \int_1^3 \int_0^{3-y} f(x,y)\,dx\,dy$

Sketch D and reverse the order of Integration.



$$\iint_D f(x,y)\,dA = \boxed{\int_0^2 \int_{x/2}^{3-x} f(x,y)\,dy\,dx}$$