Section 15.2 - Iterated Integrals

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★ In practice we do not evaluate single integrals by using the definition - we use fundamental Theorem of Calculus (FTC)

For a fixed x on $R=[a,b]\times[c,d]$ we compute the area under $f(x,y)$ above $[c,d]$:
$$A(x)=\int_c^d f(x,y)\,dy$$

Suming up the areas as x varies over $[a,b]$ is the same as integrating A with respect to x:
$$\int_a^b A(x)\,dx=\int_a^b\left[\int_c^d f(x,y)\,dy\right]dx=\underbrace{\int_a^b\int_c^d f(x,y)\,dy\,dx}_{\text{Iterated Integral}}$$

Example Evaluate:

(a) $\displaystyle\int_0^3\int_1^2 x^2y\,dy\,dx$

$$=\int_0^3 \frac{x^2y^2}{2}\Big|_1^2\,dx$$
$$=\int_0^3 \frac{3}{2}x^2\,dx$$
$$=\frac{1}{2}x^3\Big|_0^3=\boxed{27/2}$$

(b) $\displaystyle\int_1^2\int_0^3 x^2y\,dx\,dy$

$$=\int_1^2 \frac{x^3y}{3}\Big|_0^3\,dy$$
$$=\int_1^2 9y\,dy$$
$$=\frac{9y^2}{2}\Big|_1^2=\boxed{27/2}$$

Fubini's Theorem If f is continuous on $R=[a,b]\times[c,d]$ then
$$\iint_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx\,dy$$

Counter Example: $f(x,y)=\dfrac{x^2-y^2}{(x^2+y^2)^2}$ on $R=[0,1]\times[0,1]$

★ f is continuous on all R except at $(0,0)$ - the problem

$$\int_0^1\int_0^1\frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx=\int_0^1\frac{y}{x^2+y^2}\Big|_0^1\,dx=\int_0^1\frac{1}{1+x^2}\,dx=\arctan(1)=\boxed{\pi/4}$$

Not the same!

$$\int_0^1\int_0^1\frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy=\int_0^1\frac{-x}{x^2+y^2}\Big|_0^1\,dy=\int_0^1\frac{-1}{1+y^2}\,dy=-\arctan(1)=\boxed{-\pi/4}$$

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Example Evaluate $\iint_R y\sin(xy)\,dA$ where $R = [1,2]\times[0,\pi]$

★ Integrate wrt x first: $= \int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy$

Note reverse order is harder requiring integration by parts

$= \int_0^{\pi} -y\dfrac{\cos(xy)}{y}\Big|_1^2\,dy = \int_0^{\pi} -\cos(2y)+\cos(y)\,dy$

$= -\dfrac{\sin(2y)}{2} + \sin(y)\Big|_0^{\pi} = \boxed{0}$

- Double Integral as product of 2 Single Integrals:

$$\iint_R g(x)\cdot h(y)\,dA = \int_a^b g(x)\,dx\cdot\int_c^d h(y)\,dy$$

- Integration Review:

① $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ all $n \neq -1$

② $\int x^{-1}\,dx = \ln|x| + C$

③ $\int e^x\,dx = e^x + C$

④ $\int \dfrac{1}{1+x^2}\,dx = \arctan(x) + C$

⑤ $\int \sin x\,dx = -\cos(x) + C$

⑥ $\int \cos x\,dx = \sin(x) + C$

⑦ $\int \sec^2 x\,dx = \tan(x) + C$

⑧ $\int \sec x\tan x\,dx = \sec(x) + C$

- U-Substitution: $\int g'(x)f'(g(x))\,dx = \int f'(u)\,du = f(g(x)) + C$

$u = g(x)$ $du = g'(x)\,dx$

- Integration by Parts: $\int u\,dv = uv - \int v\,du$

- Change of Coords: $\int_a^b f(x)\,dx = \int_c^d f(u)\cdot u'\,du$

where $a = u(c)$ and $b = u(d)$

- Trig Substitution: $\sin^2 x + \cos^2 = 1$ $\tan^2 x + 1 = \sec^2 x$

$\sin^2 x = \dfrac{1-\cos(2x)}{2}$ $\cos^2 x = \dfrac{1+\cos(2x)}{2}$

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• Extra Examples:

#19. $\iint_R x\sin(x+y)\,dA$, $R=[0,\pi/6]\times[0,\pi/3]$

$$= \int_0^{\pi/6}\int_0^{\pi/3} x\sin(x+y)\,dy\,dx = \int_0^{\pi/6} -x\cos(x+y)\Big|_0^{\pi/3}\,dx = \int_0^{\pi/6} -x\cos(x+\pi/3) + x\cos(x)\,dx$$

$$= -x\sin(x+\pi/3)\Big|_0^{\pi/6} + \int_0^{\pi/6}\sin(x+\pi/3)\,dx + x\sin(x)\Big|_0^{\pi/6} - \int_0^{\pi/6}\sin(x)\,dx = \boxed{-\frac{\pi}{6}+\frac{1}{2}+\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1}$$

#20. $\iint_R \frac{x}{1+xy}\,dA$, $R=[0,1]\times[0,1]$

$$= \int_0^1\int_0^1 \frac{x}{1+xy}\,dy\,dx = \int_0^1 \frac{x\ln|1+xy|}{x}\Big|_0^1\,dx = \int_0^1 \ln|1+x|\,dx \quad ①$$

① $u=\ln|1+x|$ $v=x$
$du=\frac{1}{1+x}$ $dv=1$

② $u=1+x$ $u-1=x$
$du=dx$
$u=1$ $u=2$

$$= x\ln|1+x|\Big|_0^1 - \int_0^1 \frac{x}{1+x}\,dx$$

$$= \ln(2) - \int_1^2 1-\frac{1}{u}\,du \; ② = \boxed{\ln(2)-1+\ln(2)}$$

#35. Find the average value of $f(x,y)=x^2y$ over $[-1,1]\times[0,5]$

$$f_{ave} = \frac{1}{(2)\times(5)}\int_{-1}^1\int_0^5 x^2y\,dy\,dx = \frac{1}{10}\int_{-1}^1 \frac{x^2}{2}(25)\,dx = \frac{25}{20}\frac{x^3}{3}\Big|_{-1}^1 = \frac{25}{30} = \boxed{\frac{5}{6}}$$

#38. Use symmetry to compute $\iint_R (1+x^2\sin y+y^2\sin x)\,dA$, $R=[-\pi,\pi]\times[-\pi,\pi]$

$$\iint_R 1\,dA + \iint_R x^2\sin y\,dA + \iint_R y^2\sin(x)\,dA = \iint_R 1\,dA = \boxed{4\pi^2}$$

(integral of sin from $-\pi$ to π is 0)

#39. Use WolframAlpha to compute $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx$ and $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy$.

Do your answers contradict Fubini's Theorem? Explain.

$$\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx = \boxed{\frac{1}{2}} \qquad \int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy = \boxed{-\frac{1}{2}}$$

This doesn't contradict Fubini's Theorem as $\frac{x-y}{(x+y)^3}$ is not continuous on $[0,1]\times[0,1]$.