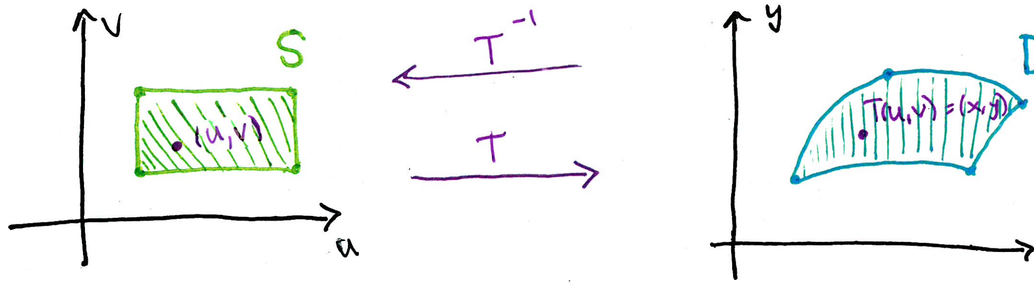
Section 15.10 - Change of Variables

★ Cylindrical and spherical coordinates are not the only coordinate systems
We can create lots of other coordinate systems.



$x = g(u,v)$
$y = h(u,v)$
$u = K(x,y)$
$v = \ell(x,y)$

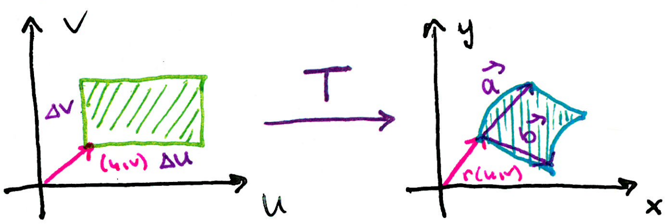
Transformation: a mapping between points (u,v) in S to its image (x,y) in D

One-to-One Transformation: if no two points have the same image.

C' Transformation: the functions $x = g(u,v)$, $y = h(u,v)$ have continous 1st order partial Derivatives.

★ Mapping Bandaries: For 1-1 C' Transformations Bandaries map to bondaries.

- Changing variables:



Goal: Approximate the blue shape with a parallelogram in terms of u, v.

★ Visit website to see demos

$$T: \vec{r}(u,v) = \langle g(u,v), h(u,v)\rangle = \langle x, y\rangle$$

$$\vec{a} = \frac{\vec{r}(u, v+\Delta v) - \vec{r}(u,v)}{\Delta v} \cdot \Delta v = \vec{r}_v \Delta v \qquad \vec{b} = \frac{\vec{r}(u+\Delta u, v) - \vec{r}(u,v)}{\Delta u} \Delta u = \vec{r}_u \Delta u$$

Parallelogram area $= |\vec{a} \times \vec{b}| = |\vec{r}_v \Delta v \times \vec{r}_u \Delta u| = |\vec{r}_u \times \vec{r}_v| \Delta v \Delta u = \left\| \begin{matrix} i & j & k \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \end{matrix} \right\| \Delta v \Delta u = K \left\| \begin{matrix} \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \end{matrix} \right\| \Delta v \Delta u$

Jacobian of T: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix} = \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u} - \dfrac{\partial y}{\partial v}\dfrac{\partial x}{\partial u}$

- Change of variables in Double Integrals:

T a C' Transformation whose Jacobian is nonzero and maps S onto R. f continous on R, T 1-1 except on bondary of S then,

$$\boxed{\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv}$$

Example Show change of variables from Cartesian to polar in a double integral gives $dA = r\,dr\,d\theta$.

$x = r\cos\theta \quad y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin\theta = r$$

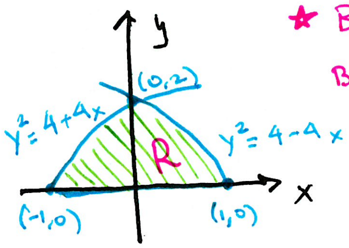
$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta \quad ■$$

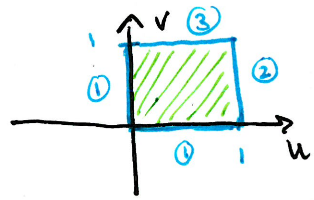
Example Use the transformation defined by $x = u^2 - v^2$ and $y = 2uv$, to evaluate $\iint_R y\,dA$ where R is bounded by the x-axis and parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 2u & 2v \\ -2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0 \text{ for } (u,v) \neq (0,0)$$

① Find $S = \text{Image}(R)$

★ Boundaries of R $\downarrow$ Boundaries of S





① $y = 0 \quad -1 \leq x \leq 1$

$0 = 2uv \quad -1 \leq u^2 - v^2 \leq 1$

$u = 0 \text{ or } v = 0 \quad 0 \leq u \leq 1 \quad 0 \leq v \leq 1$

② $y^2 = 4 - 4x \quad 0 \leq x \leq 1$, $0 \leq y \leq 2$

$4u^2v^2 = 4 - 4(u^2 - v^2)$

$u^2 = 1 \quad u = 1 \quad 0 \leq v \leq 1$

$u = -1 \quad -1 \leq v \leq 0$

③ $y^2 = 4 + 4x$

$v^2 = 1 \quad v = 1 \quad 0 \leq u \leq 1$

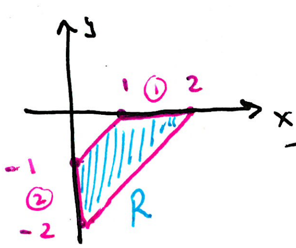
$v = -1 \quad -1 \leq v \leq 0$

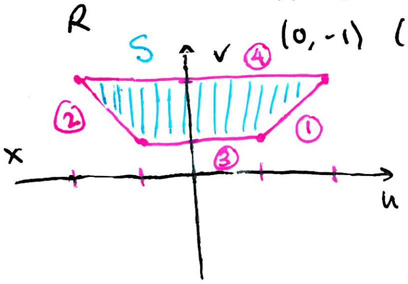
$$\iint_R y\,dA = \int_0^1\int_0^1 2uv(4u^2 + 4v^2)\,du\,dv$$

$$= \int_0^1 8\left(\tfrac{1}{4}\right)(v) + 8\left(\tfrac{1}{2}\right)v^3\,dv$$

$$= 2\left(\tfrac{1}{2}\right) + 4\left(\tfrac{1}{4}\right) = \boxed{2}$$

Example Evaluate $\iint_R e^{\left(\frac{(x+y)}{(x-y)}\right)}\,dA$ where R is the trapezoid region with vertices $(0,-1)$ $(0,-2)$ $(2,0)$ $(1,0)$.





$u = x + y \quad v = x - y$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \left|-\tfrac{1}{2}\right|$$

$x = \dfrac{u+v}{2} \quad y = \dfrac{u-v}{2}$

$$= \int_1^{2}\int_{-v}^{v} e^{u/v}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv$$

$$= \frac{1}{2}\int_1^2 v(e^1 - e^{-1})\,dv$$

$$= \boxed{\frac{1}{4}(4-1)(e^1 - e^{-1})}$$

① $y = 0 \quad 1 \leq x \leq 2$

$u = v = x$

② $x = 0 \quad -1 \leq y \leq -2$

$u = -v = y$

③ $y = x - 1 \quad 0 \leq x \leq 1$

$v = 1$

④ $y = x - 2 \quad 0 \leq x \leq 2$

$v = 2$

Section 15.10 - Change of Variables

- Change of Variables for Triple Integrals:

$$T: \quad x = g(u,v,w) \qquad y = h(u,v,w) \qquad z = k(u,v,w)$$

Jacobian of T: $\displaystyle \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Example Derive the formula for Triple integrals in spherical coordinates.

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$

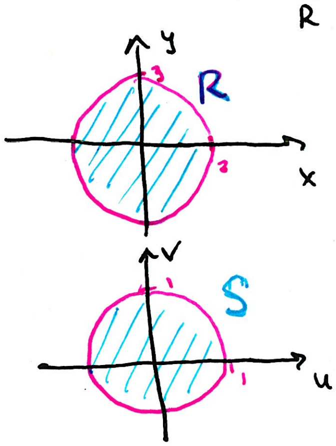
$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ -\rho\sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & 0 \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi \end{vmatrix} = -\rho^2\sin^3\varphi\cos^2\theta - \rho^2\sin^3\varphi\sin^2\theta + \cos\varphi\left(-\rho^2\cos\varphi\sin\varphi\sin^2\theta - \rho^2\cos\varphi\sin\varphi\cos^2\theta\right)$$

$$= -\rho^2\sin^3\varphi + \cos\varphi\left(-\rho^2\cos\varphi\sin\varphi\right) = \boxed{-\rho^2\sin\varphi}$$

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi \quad \blacksquare$$

- Extra Examples:

#17 Evaluate $\iint_R x^2\,dA$, where R is the region bounded by $9x^2 + 4y^2 = 36$, use $x = 2u$, $y = 3v$.



$$9\cdot 4u^2 + 4\cdot 9v^2 = 36$$

$$u^2 + v^2 = 1$$

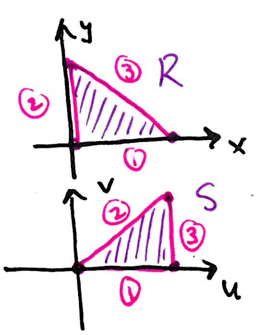
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

$$\iint_R x^2\,dA = \int_0^{2\pi}\int_0^1 (2u)^2 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 24r^3\cos^2\theta\,dr\,d\theta$$

$$= 24\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\int_0^{2\pi}(1 + \cos(2\theta))\,d\theta$$

$$= \boxed{6\pi}$$

#28 Let f be continuous on $[0,1]$ and let R be the triangle with vertices $(0,0)$ $(0,1)$ $(1,0)$. Show that $\displaystyle\iint_R f(x+y)\,dA = \int_0^1 u f(u)\,du.$



$u = x + y$

$v = y$

$x = u - v$

Check boundaries:

① $y = 0 \quad 0 \le x \le 1$

$0 \le u \le 1 \quad v = 0$

② $x = 0 \quad 0 \le y \le 1$

$u = v$

③ $x + y = 1 = u$

$$= \iint_S f(u) \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} du\,dv$$

$$= \int_0^1\int_0^u f(u)\,dv\,du = \int_0^1 u f(u)\,du \quad \blacksquare$$